ASC Course Final

ASC-TMP final examination for the Toric Varieties course

You have time until Friday, 13. May 2011.

You can answer the following questions either manually or by a Sage computation. In either case, you are to work on your own and show your work. You can write down your answers in this worksheet and share it with user `vbraun`, create a PDF file and email it to `vbraun@stp.dias.ie` with a Subject: that includes your account name, or both.

1. Intersection Theory

Consider the following 4-dimensional toric variety:

```
lp = LatticePolytope(matrix([[1,0,0,0],[0,1,0,0],[0,0,1,0],
[0,0,0,1],[-1,-1,-3,-3]]).transpose())
X = ToricVariety(FaceFan(lp), coordinate_names='x+')
X.inject_variables()
```

```
Defining x0, x1, x2, x3, x4
```

Q1: List the torus orbits (for example, by the their corresponding cones) that consist of non-smooth points in X.

Q2: The singular set X_{sing} is the set of non-smooth points in X. It is the union of the torus orbits of the previous question. Write X_{sing} in terms of homogeneous coordinates.

Q3: Let $D = \{x_3 = 0\}$ be one of the Cartier divisors of X_{sing}. In terms of homogeneous coordinates, what is the intersection $X_{\text{sing}} \cap D$? The intersection consists of how many points?

Q4: Compute the intersection number in the Chow group $A_*(X)$

2. Crepant Resolutions

A resolution of singularities is called *crepant* if the canonical class "stays the same".

In general this means that if $f : Y \to X$ is the resolution map, then the pullback $f^*(K_X) = K_Y$ of the canonical class on X equals the canonical class on Y.

For a Gorenstein toric variety, the canonical class K is given by a integral support function φ_K modulo everywhere linear functions. A resolution of Gorenstein toric varieties coming from a subdivision of the fan is crepant if the support function φ_{K_X} equals the support function φ_{K_Y} up to an everywhere linear function.

Q5: Check that the toric variety X defined above is Gorenstein.

Here is one possible resolution of X into a smooth four-fold Y:

```
resolved_fan = X.fan().subdivide(new_rays=[(0,0,-1,-1)])
Y = ToricVariety(resolved_fan, coordinate_names='y0, y1, y2, y3,
y4, t')
Y.inject_variables()
```

```
[X.is_smooth(), Y.is_smooth()]
```

```
Defining y0, y1, y2, y3, y4, t
[False, True]
```

Q6: Is this resolution crepant?

A Calabi-Yau subvariety $V(x_0^9 + x_1^9 + x_4^9 + x_2^3 + x_3^3) \subset X$ is singular because it necessarily intersects X_{sing}.

Consider $Z = V\big((y_0^9 + y_1^9 + y_4^9)t^3 + y_2^3 + y_3^3\big) \subset Y$. Setting $t = 1$ and $y_i = x_i$ recovers the singular Calabi-Yau variety in X. Because the resolution is crepant, the homology class of Z cancels the canonical class in the adjunction formula $K_Z = (K_Y + [Z])|_Z$ in the same way as they cancelled on X. Hence Z is Calabi-Yau, too. But Z might still be singular because of the form of its defining equation.

Q7: Show that Z is, in fact, smooth. Therefore, Z is a smooth Calabi-Yau threefold.

3. Hilbert Polynomials

Consider the divisor $D = V(y_0) + V(y_2)$

Q8: Compute the number of sections of $\mathcal{O}(nD)$, that is, the dimension of the linear space of sections $\Gamma($, for $n \in \{-10, -9, \ldots, 9, 10\}$

Q9: Find a quartic polynomial $p_4(n) = \dim \Gamma(Y, \mathcal{O}(nD))$ for $n \in \{0, 1, \ldots, 9, 10\}$.