Sierpinski Triangle
A fractal constructed with Pascal's triangle

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Chapter 1

Sierpinski triangle

This small report tries to demonstrate that the construction of Sierpinski triangle which uses the binomial coefficient from Pascal's triangle is a fractal.

It was made in addition to the report on Pascal's triangle which can be found at :

`http://bit.ly/pascal2014`

1.1 Construction

Sierpinski triangle is a fractal which is simple to create. It has the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles [1] as seen in figure 1.1.

But it can also be created using Pascal's triangle by coloring all the odd numbers as shown in figure 1.2. It is this construction that we are going to study.

1.2 Definition

Definition 1 (Fractals[2]). *Geometric fractals are fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole. Authorities disagree on the exact definition of fractals, but most usually elaborate on the basis ideas of self-similarity and an unusual relationship with the space a fractal is embedded in.*

But we are going to focus on the fractal's property of self-similarity at different scale to prove that Sierpinski triangle is a fractal.

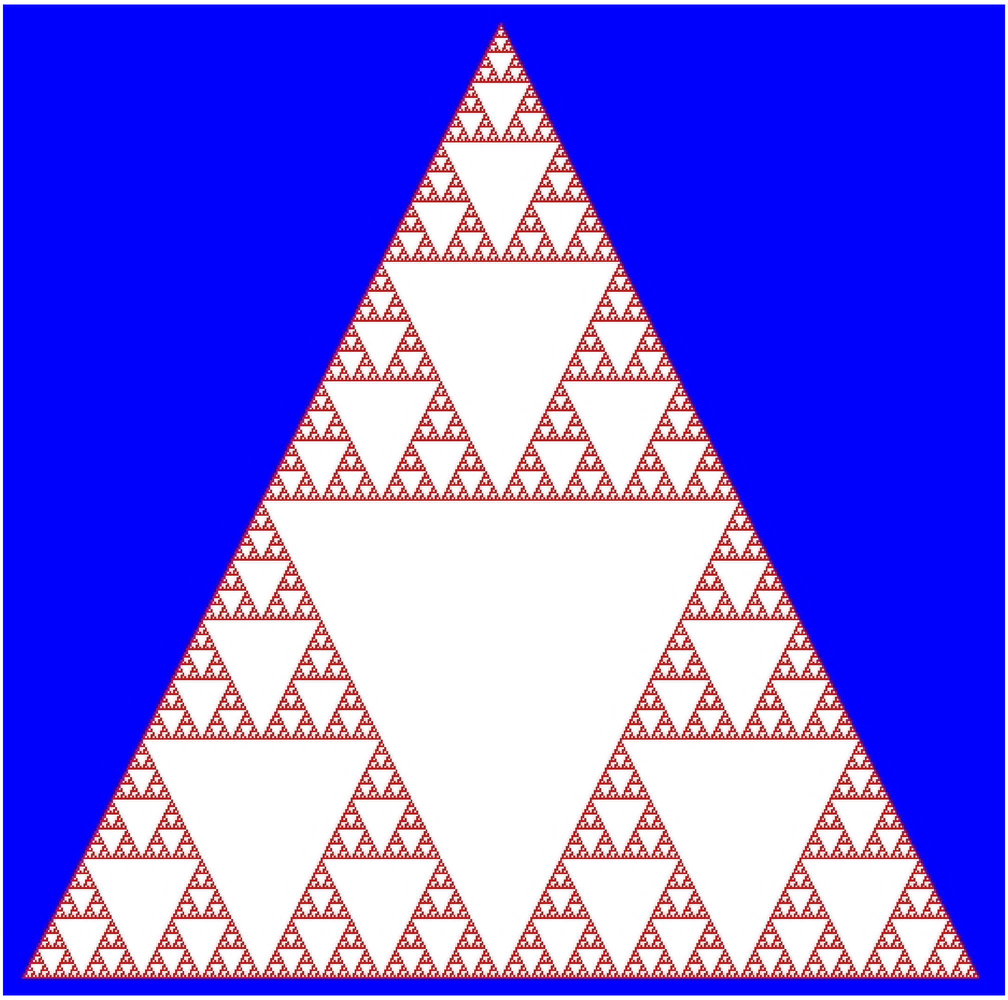

Figure 1.1: A Sierpinski triangle

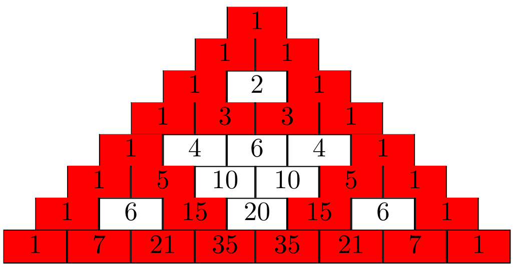

Figure 1.2: Pascal's triangle use for the creation of Sierpinski triangle

Chapter 2

Proof

We want to proove that Sierpinski triangle is a fractal using the self-similarity property. We are going to proceed in 3 steps :

1. All $\binom{2^n}{k}, n \in \mathbb{N}, 0 < k < 2^n$ are even.
2. Such a property will lead to the formation of a triangular area.
3. Sierpinski triangle can be divided into 4 triangles, 3 of them congruent and which can be divided in the same way as the former triangle.

2.1 Step 1 : $n \in \mathbb{N}, 0 < k < 2^n : \binom{2^n}{k} \equiv 0 \pmod 2$

We are first going to prove the following lemma :

Lemma 1 (Parity of $\binom{2^n-1}{k}$)**.** $(n, k) \in \mathbb{N}$

$$0 \leq k \leq 2^n - 1 \Rightarrow \binom{2^n - 1}{k} \equiv 1 \pmod 2 \tag{2.1}$$

To do so, let C the functions that maps $\mathbb{N} - \{0\}$ onto $\mathbb{N}$ where :

$$n = 2^\alpha \times k, k \equiv 1 \pmod 2 \tag{2.2}$$

$$C(n) = \alpha \tag{2.3}$$

In words, $C(n)$ is the maximum number of time n can be divided by 2 and still be an integer.
This function has three properties wich are prooved in the appendix B on page 11.

Property 1. *Let a and b natural numbers :*

$$C(ab) = C(a) + C(b) \tag{2.4}$$

Property 2. *Let a and b natural numbers and $\frac{a}{b} \in \mathbb{N}$:*

$$C(a) = C(b) \Rightarrow \frac{a}{b} \equiv 1 \pmod 2 \tag{2.5}$$

Property 3. *Let a and b natural numbers :*

$$a < 2^n \Rightarrow C(a) < n \tag{2.6}$$

Proof by induction. We are going to prove that :

$$\begin{cases} n \in \mathbb{N} \\ k \in \mathbb{N}, 0 \leq k \leq 2^n - 1 \end{cases} \Rightarrow \binom{2^n - 1}{k} \equiv 1 \pmod 2 \tag{2.7}$$

Base case. With lemma 2 we have $\binom{2^n-1}{0} = 1 \equiv 1 \pmod 2$, for every $n \in \mathbb{N}$.

However with construction 2 we have : $\binom{2^n-1}{0} = \frac{(2^n-1)!}{0!(2^n-1-0)!} = 1$. Thus, for $k = 0$ and all whole n we have :

$$(2^n - 1)! = k!(2^n - 1 - k)! \Rightarrow C([2^n - 1]!) = C(k!(2^n - 1 - k)!) \tag{2.8}$$

Inductive step Let a rank $p \in \mathbb{N}, 0 \le p < 2^n - 1$ for which for all whole n we have :

$$C([2^n - 1]!) = C(p!(2^n - 1 - p)!) \tag{2.9}$$

Let K such as $K \equiv 1 \pmod 2$ and $p + 1 = 2^{C(p+1)} \times K$.

But $p < 2^n - 1$, thus $p + 1 < 2^n$ and with property 3 we have $C(p + 1) < n$ and $n - C(p + 1) > 0$.

Then :

$$A : \begin{cases} n - C(p+1) > 0 \\ K \equiv 1 \pmod 2 \\ 2^n - (p+1) = 2^n - 2^{C(p+1)}K \end{cases} \tag{2.10}$$

$$A \Rightarrow \begin{cases} 2^{n-C(p+1)} \equiv 0 \pmod 2 \\ 2^{n-C(p+1)} - K \equiv 1 \pmod 2 \\ 2^n - (p+1) = 2^{C(p+1)}\left(2^{n-C(p+1)} - K\right) \end{cases} \Rightarrow C(2^n - (p+1)) = C(p+1) \tag{2.11}$$

However with property 1 of C, we have :

$$C\big([p+1]![2^n - 1 - (p+1)]!\big) = C([p+1]!) + C([2^n - p - 2]!) \tag{2.12}$$

$$C([p+1]!) = C(p!) + C(p+1) \tag{2.13}$$

$$C([2^n - p - 2]!) = C([2^n - p - 1]!) - C(2^n - p - 1) \tag{2.14}$$

$$= C([2^n - p - 1]!) - C(p+1) \tag{2.15}$$

Thus :

$$C([p+1]![2^n - 1 - (p+1)]!) = C(p!) + C(p+1) + C([2^n - p - 1]!) - C(p+1) \tag{2.16}$$

$$= C(p!) + C([2^n - p - 1]!) = C\big(p!(2^n - p - 1)!\big) \tag{2.17}$$

Thus with the induction hypothesis (equation 2.9), for all whole n, we have :

$$C([2^n - 1]!) = C\big([p+1]![2^n - 1 - (p+1)]!\big) \tag{2.18}$$

Since the basis and inductive step have been performed, by mathematical induction, property 2 of C, construction 2 and property 5 :

$$\begin{cases} k \in \mathbb{N}, 0 \le k \le 2^n - 1 \\ n \in \mathbb{N} \end{cases} \Rightarrow C([2^n - 1]!) = C(k!(2^n - 1 - k)!) \tag{2.19}$$

$$\Rightarrow \frac{(2^n - 1)}{k!(2^n - 1 - k)} = \binom{2^n - 1}{k} \equiv 1 \pmod 2 \tag{2.20}$$

But we need to check the parity of $\binom{2^n}{k}$.

$$B : \begin{cases} n \in \mathbb{N} \\ k \in \mathbb{N}, 0 < k < 2^n \end{cases} \Rightarrow \binom{2^n}{k} = \binom{2^n - 1}{k - 1} + \binom{2^n - 1}{k} \equiv 1 + 1 \pmod 2 \equiv 0 \pmod 2 \tag{2.21}$$

We thus have (with lemmas 2 and 3) :

Property 4 (Parity of $\binom{2^n}{k}$)**.** $(n, k) \in \mathbb{N}^2$

$$\begin{cases} n \in \mathbb{N} \\ k \in \mathbb{N}, 0 < k < 2^n \end{cases} \Rightarrow \begin{cases} \binom{2^n}{k} \equiv 0 \pmod 2 \\ \binom{2^n}{0} = \binom{2^n}{2^n} \equiv 1 \pmod 2 \end{cases} \tag{2.22}$$

□

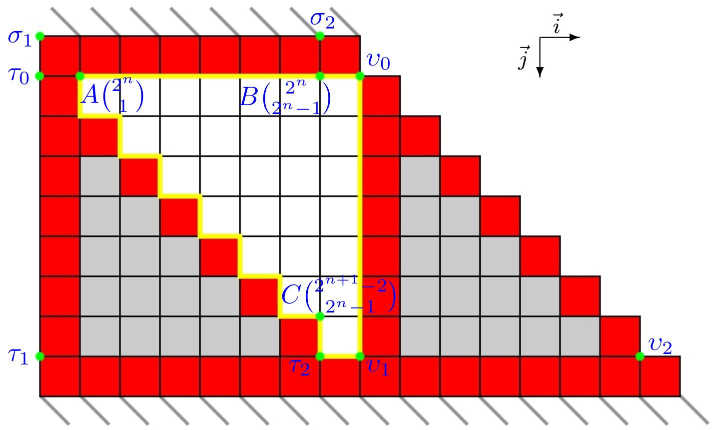

Figure 2.1: Close-up of Sierpinski triangle (annotated)

2.2 Step 2 : Formation of a triangle

Let a cartesian coordinate system $(O, \vec{i}, \vec{j})$, where O is the upper vertex of Sierpinski's triangle, and $\vec{i}$ and $\vec{j}$ are define as seen in the figure 2.1.

We are also going to associate to each point with whole coordinates $\binom{x}{y}$, $0 \leq y \leq x$, a color. The point of coordinates $\binom{x}{y}$ will be of color "even"[1] if the binomial coefficient $\binom{x}{y}$ is even, otherwise the point is considered as "odd"[2].

We are now going to prove that the property 4 in step 1 leads to the formation of a triangle, which would have for vertex $A\binom{2^n}{1}$, $B\binom{2^n}{2^n-1}$ and $C\binom{2^{n+1}-2}{2^n-1}$; $n \in \mathbb{N}$.

To do so, we are going to study the yellow area represented on figure 2.1. To prove that it is a triangle we are going to study the following properties :

- It has 3 sides.
- All the points in the area are "even".
- All the points directly next to[3] the area are "odd".

On figure 2.1, the yellow area doesn't look as a triangle. However it is important to remind that Sierpinski triangle is only a fractal if the Pascal's triangle, used to create it, had an infinity of lines. If there is an infinity of lines, when we will be looking at the whole triangle, the space between colored points will be nil, as we would be seing an infinity of colored points in a limited area (usually the area of a sheet of paper[4]).

Thus aligned colored points would look like a line (as there will not be any space between them). So to proved our area has 3 sides, we are going to prove that the top, right and left sides of our area are lines (AB, AC and BC). And to prove that the points next to the area are "odd" we are going to check the color of the lines next to the sides of our triangles.

But first, we are going to prove recursively that for $n \in \mathbb{N}$:

$$\begin{cases} \alpha \in \mathbb{N}, 0 \leq \alpha < 2^n \\ k \in \mathbb{N}, \alpha < k < 2^n \end{cases} \Rightarrow \begin{cases} \binom{2^n+\alpha}{\alpha} \equiv 1 \pmod 2 \\ \binom{2^n+\alpha}{2^n} \equiv 1 \pmod 2 \\ \binom{2^n+\alpha}{k} \equiv 0 \pmod 2 \end{cases} \tag{2.23}$$

Recursive proof. Let $n \in \mathbb{N}$.

[1] Represented as white squares on figure 2.1
[2] Represented as red squares on figure 2.1
[3] Above, below, right, left (not diagonaly)
[4] As in figure 1.1

Base case. Let $\alpha = 0$. Then : With property 4 we have :

$$\begin{cases} n \in \mathbb{N} \\ k \in \mathbb{N}, 0 < k < 2^n \end{cases} \Rightarrow \begin{cases} \binom{2^n+\alpha}{\alpha} = \binom{2^n}{0} \equiv 1 \pmod 2 \\ \binom{2^n+\alpha}{2^n} = \binom{2^n}{2^n} \equiv 1 \pmod 2 \\ \binom{2^n+\alpha}{2^n} = \binom{2^n}{k} \equiv 0 \pmod 2 \end{cases} \tag{2.24}$$

Thus the basic step has been performed.

Inductive step. Let $p \in \mathbb{N}$ such as $0 \leq p < 2^n - 1$ and :

$$k \in \mathbb{N}, p < k < 2^n \Rightarrow \begin{cases} \binom{2^n+p}{p} \equiv 1 \pmod 2 \\ \binom{2^n+p}{2^n} \equiv 1 \pmod 2 \\ \binom{2^n+p}{k} \equiv 0 \pmod 2 \end{cases} \tag{2.25}$$

So for $p+1$ we have ($k \in \mathbb{N}$) :

$$p < 2^n - 1 \Rightarrow p < p+1 < 2^n \Rightarrow \binom{2^n+p+1}{p+1} = \binom{2^n+p}{p} + \binom{2^n+p}{p+1} \equiv 1+0 \pmod 2 \tag{2.26}$$

$$p < 2^n - 1 \Rightarrow p < 2^n - 1 < 2^n \Rightarrow \binom{2^n+p+1}{2^n} = \binom{2^n+p}{2^n-1} + \binom{2^n+p}{2^n} \equiv 0+1 \pmod 2 \tag{2.27}$$

$$p+1 < k+1 < 2^n \Rightarrow p < k < 2^n \Rightarrow \binom{2^n+p+1}{k+1} = \binom{2^n+p}{k} + \binom{2^n+p}{k+1} \equiv 0+0 \pmod 2 \tag{2.28}$$

Which can be sum up as :

$$\begin{cases} k \in \mathbb{N} \\ p+1 < k+1 < 2^n \end{cases} \Rightarrow \begin{cases} \binom{2^n+p+1}{p+1} \equiv 1 \pmod 2 \\ \binom{2^n+p+1}{2^n} \equiv 1 \pmod 2 \\ \binom{2^n+p+1}{k} \equiv 0 \pmod 2 \end{cases} \tag{2.29}$$

Since the basis and the inductive steps have been performed, by mathematical induction ($\forall n \in \mathbb{N}$) :

$$\begin{cases} \alpha \in \mathbb{N}, 0 \leq \alpha < 2^n \\ k \in \mathbb{N}, \alpha < k < 2^n \end{cases} \Rightarrow \begin{cases} \binom{2^n+\alpha}{\alpha} \equiv 1 \pmod 2 \\ \binom{2^n+\alpha}{2^n} \equiv 1 \pmod 2 \\ \binom{2^n+\alpha}{k} \equiv 0 \pmod 2 \end{cases} \tag{2.30}$$

With this done we can now prove the 3 property of our area.
Let's consider the top side :

- All the points of the top side 2^n as y-coordinates and there x-coordinate vary from 1 to $2^n - 1$, thus they are aligned. However the points of AB also share this property, thus AB is indeed a side of the triangle.
- The points above AB have the coordinates $\binom{2^n-1}{k}, 0 < k < 2^n$. However in step 1 (lemma 1) we proved that these binomial coefficient were odd. Thus, directly above AB, there are only "odd" points.

Let's consider the right side :

- All the points of the right side have $2^n - 1$ as x-coordinates and there y-coordinate vary from 2^n to $2^n + 2^n - 1 = 2^{n+1} - 1$, thus they are aligned. However the points of BC also share this property, thus BC is indeed a side of the triangle.
- The points right of BC have the coordinates $\binom{2^n+\alpha}{2^n}, 0 \geq \alpha < 2^n - 1$. However we juste proved that for an α in those conditions : $\binom{2^n+\alpha}{2^n} \equiv 1 \pmod 2$. Thus, directly right of BC, there are only "odd" points.

Let's consider the left side :

- All the points of the left side have $\binom{2^n+\alpha}{\alpha+1}, 0 \geq \alpha < 2^n - 1$ as coordinates, thus they are on all on the line of equation $y = 2^n - 1 + x$. However the points of AC share this property, thus AC in indeed a side of the triangle.
- The points left of AC have the coordinates $\binom{2^n+\alpha}{\alpha}, 0 \geq \alpha < 2^n - 1$. However we juste proved that for an α in those conditions : $\binom{2^n+\alpha}{\alpha} \equiv 1 \pmod 2$. Thus, directly left of AC, there are only "odd" points.

Plus, directly below C, the point of coordinates $\binom{2^{n+1}-1}{2^n-1}$ is "odd," as we proved with lemma 1.

With the recursive proof we just did, if we divide the area of our triangle into lines, then[5] we can say that every points on each line of triangle ABC are "even", thus every points in the area of the triangle are "even".

Thus we have indeed the formation of an area which is a triangle. $\square$

2.3 Step 3 : Self-similarity

To finish our proof, we need to prove the self-similarity of the Sierpinski's triangle.

We are going to divide our Sierpinski's triangle into 4 areas[6] : $O\sigma_1\sigma_2$, $\tau_0\tau_1\tau_2$, $\upsilon_0\upsilon_1\upsilon_2$, and ABC.

First of all, let's prove that the 3 areas $O\sigma_1\sigma_2$, $\tau_0\tau_1\tau_2$, and $\upsilon_0\upsilon_1\upsilon_2$ are congruent.

With the figure 2.1 we can work out the coordinates of all the points :

$$O\binom{0}{0} \qquad \tau_0\binom{2^n}{0} \qquad \upsilon_0\binom{2^n}{2^n} \tag{2.31}$$

$$\sigma_1\binom{2^n-1}{0} \qquad \tau_1\binom{2^{n+1}-1}{0} \qquad \upsilon_1\binom{2^{n+1}-1}{2^n} \tag{2.32}$$

$$\sigma_2\binom{2^n-1}{2^n-1} \qquad \tau_2\binom{2^{n+1}-1}{2^n-1} \qquad \upsilon_1\binom{2^{n+1}-1}{2^{n+1}-1} \tag{2.33}$$

We can then work out some lenghts :

$$O\sigma_1 = \tau_0\tau_1 \quad = \upsilon_0\upsilon_1 = 2^n - 1 \tag{2.34}$$

$$\sigma_1\sigma_2 = \tau_1\tau_2 \quad = \upsilon_1\upsilon_2 = 2^n - 1 \tag{2.35}$$

$$0\sigma_2{}^2 = \tau_0\tau_2{}^2 = \upsilon_0\upsilon_2{}^2 = 2 \times (2^n - 1)^2 \tag{2.36}$$

So the 3 areas have the same dimensions. To check if they are congruent we also need to check the color of their points.

With lemmas 2 and 3, we know that the segments $O\sigma_1$, $O\sigma_2$, $\tau_0\tau_1$, and $\upsilon_0\upsilon_2$ only contains "odd" points.

With proved that next to segments AC and BC there were only odd points. However $\tau_1\tau_2$ is next to AC and $\upsilon_0\upsilon_1$ next to BC. Thus $\tau_1\tau_2$ and $\upsilon_0\upsilon_1$ only contains "odd" points.

So the left and right sides of the areas $O\sigma_1\sigma_2$, $\tau_0\tau_1\tau_2$, and $\upsilon_0\upsilon_1\upsilon_2$ only contains "odd" points and are thus congruent. However, with construction 1 we have :

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \equiv \binom{n}{k} + \binom{n}{k+1} \pmod 2 \tag{2.37}$$

Thus we know that the color (the parity of the corresponding binomial coefficient) of a point only depends of the color of the points above it.

All 3 areas have the same starting color : $O\sigma_1$, $\tau_0\tau_1$ and $\upsilon_0\upsilon_1$ are "odd". However with what we just proved, we can say that the colors of the point in one area only depends on the color of these starting points. Thus because the 3 areas have the same starting points, they will form the same pattern.

Thus these 3 areas are congruent.

[5]With $\binom{2^n+\alpha}{k} \equiv 0 \pmod 2$

[6]See figure 2.1

So at a given 2^n number of lines, Sierpinski's triangle can be divided into 3 congruent triangles [7] and an "even" triangle[8]. However, this property is also true at 2^{n-1}, which means that the top triangle[9] can also be subdivided into 3 congruent triangles and an "even" triangle.

This means that when n tends to $+\infty$, then Sierpinski's triangle can be divided into smaller chunks which can themselves be divided into smaller chunks. And at all levels we would be looking at the same picture : an "even" triangle surrounded by congruent triangles.

Thus Sierpinski's triangle respect the self-similarity property of the fractals.

Thus Sierpinski's triangle is a fractal. $\square$

[7] $O\sigma_1\sigma_2$, $\tau_0\tau_1\tau_2$, and $\upsilon_0\upsilon_1\upsilon_2$

[8] ABC

[9] $O\sigma_1\sigma_2$ in our example

Appendix A

Properties of the binomial coefficients

Most of the proof of these properties can be found in the original report on Pascal's triangle[1].

Construction 1 (Recursive construction)**.** $n \in \mathbb{N}$

$$\binom{0}{0} = 1 \tag{A.1}$$

$$\forall p \in \mathbb{Z}, p \neq 0 : \binom{0}{p} = 0 \tag{A.2}$$

$$\forall k \in \mathbb{Z} : \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \tag{A.3}$$

Construction 2 (In terms of n and k)**.** $(n, k) \in \mathbb{N}^2$

$$\forall k, 0 \leq k \leq n : \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{A.4}$$

Lemma 2. *Let $n \in \mathbb{N}$, then :* $\binom{n}{0} = 1$

Lemma 3. *Let $n \in \mathbb{N}$, then :* $\binom{n}{n} = 1$

Property 5. $\forall (n, k) \in \mathbb{N}^2, 0 \leq k \leq n : \binom{n}{k} \in \mathbb{N}$

[1] See `http://bit.ly/pascal2014`

Appendix B

Proof of the properties of function C on page 4

Proof of property 1 on page 4 : $C(ab) = C(a) + C(b)$. Let $(a, b) \in (\mathbb{N} - \{0\})^2$.
By definition of function C :

$$a = 2^\alpha \times k_a, k_a \equiv 1 \pmod 2 \Rightarrow C(a) = \alpha \tag{B.1}$$

$$b = 2^\beta \times k_b, k_b \equiv 1 \pmod 2 \Rightarrow C(b) = \beta \tag{B.2}$$

Thus $ab = 2^{\alpha+\beta} \times k_a k_b$ and $k_a \times k_b \equiv 1 \times 1 \pmod 2 \equiv 1 \pmod 2$.
Thus :

$$C(ab) = \alpha + \beta = C(a) + C(b) \qquad \square$$

Proof of property 2 on page 4 : $C(a) = C(b) \Rightarrow \frac{a}{b} \equiv 1 \pmod 2$. Let $(a, b) \in (\mathbb{N} - \{0\})^2$ and $\frac{a}{b} \in \mathbb{N}$.
Let q such as $q = \frac{a}{b}$. Then :

$$C(a) = C(b) \Rightarrow \begin{cases} a = 2^\alpha \times k_a, k_a \equiv 1 \pmod 2 \\ b = 2^\alpha \times k_b, k_b \equiv 1 \pmod 2 \end{cases} \Rightarrow q = \frac{a}{b} = \frac{k_a}{k_b} \Rightarrow k_a = q k_b \tag{B.3}$$

q is either even or odd. If q even then :

$$q \equiv 0 \pmod 2 \Rightarrow q k_b \equiv 0 \pmod 2 \Rightarrow k_a \equiv 0 \pmod 2 \tag{B.4}$$

But k_a is odd, thus q cannot be even, and q is odd. Thus $\frac{a}{b} \equiv 1 \pmod 2$.
Thus :

$$C(a) = C(b) \Rightarrow \frac{a}{b} \equiv 1 \pmod 2 \qquad \square$$

Proof of property 3 on page 4 : $a < 2^n \Rightarrow C(a) < n$. Let $(a, n) \in (\mathbb{N} - \{0\})^2$.
We are going to do a proof by contradiction :
Let a as $a < 2^n$ and $a = 2^{C(a)} \times k$ with $k \equiv 1 \pmod 2$.
Suppose that $C(a) \geq n$, then :

$$\begin{cases} 2^{C(a)} \geq 2^n \\ k \equiv 1 \pmod 2 \Rightarrow k \geq 1 \end{cases} \Rightarrow 2^{C(a)} \times k \geq 2^n \times k \geq 2^n \tag{B.5}$$

$$\Rightarrow a \geq 2^n \tag{B.6}$$

However $a < 2^n$, then this contradicts our assumtion so we are forced to conclude that :

$$a < 2^n \Rightarrow C(a) < n \qquad \square$$

References

[1] Sierpinski triangle. en.wikipedia.org/wiki/Sierpinski_triangle.

[2] Fractal. en.wikipedia.org/wiki/Fractal.