Pascal's Triangle
A journey from combinatorics to fractals.

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Motivation. You surely have already seen this triangle somewhere during your studies. It could have been in math class while working on probabilities or more precisely combinatorics. Or maybe you had to expand something like $(a+b)^n$. Or maybe you saw it out of the classroom, as a disguised form of art, as a fractal.

Or at least, this is how I met *Pascal's triangle*. But because in class we always used it in some mysterious ways, like in a binomial experiment, I thought it would be a good idea to look under the hood and try to figure out why we alleged certain uses and property to *Pascal's triangle* and its binomial coefficients.

Wherever you saw it, there is a high chance you will stumble upon it again in places you would not expect it, like when working out the sum of the first natural numbers.

Focus. My goal in this report is not to catalogue all the places you can encounter *Pascal's triangle* nor to do an exhaustive list of all its properties. It has indeed so many of them that if someone finds another it will only interest himself [1].

My goal is to give you an insight of where it comes from and most importantly, prove some of the most common properties that are alleged to it.

Method. To fulfill this goal, I will begin by telling you the story of *Pascal's triangle* and then prove the coherence between the 3 most common ways of constructing it.

Then I am going to study its use in combinatorics and show some of the most known properties of the triangle.

But writing about *Pascal's triangle* without mentioning its counterpart Sierpinski triangle would be a mistake. So I will end this report with fractals.

So let's get started in this journey from combinatorics to fractals.

Chapter 1

Construction of Pascal's triangle

1.1 Historical construction

Pascal's triangle was known a long time ago (up to 2 centuries before BC), but the work we got back is sparse and in fragments [2]. Some mathematicians between the 11^{nt} and the 13^{nt} century did similar work as Blaise Pascal on the triangle in Iran and China [2], which explain why it as different name in these regions[1]. However in the Western world we attribute Pascal's name to the triangle[2] as he wrote a treatise in 1654[3] on the subject [2].

He wrote his treatise *Traité du triangle arithmétique* after a correspondance he had with Fermat to solve a probability problem. It his treatise the triangle was presented along a series of properties similar to some of which will be presented here [3].

Problem of points or Division of the stakes [4]. The game consists of a series of rounds where the players have equal chances of winning each round. If one player has won a certain number of rounds determined beforehand, the game stops and this player recieve the prize pot (at which both players contributed equaly).

The Chevalier de Méré asked Pascal how should the stakes be divided if the game was interrupted before one of the two players had won.

Example. The score after a few rounds is 2/1 and the players wanted to go up to 3 winning rounds. However the game is interrupted. For the division of the stakes to be fair, it should take into account how many winning rounds was needed for each player to reach the threshold of 3. Because the first player only needs 1 more round and the second 2, the first player should receive a higher share. This share should be proportional to the odds the player had to win the game.

In this case, the second player would have had to win 2 consecutive rounds to win the game. The probability of winning those rounds is $0.5^2 = 0.25$. So in our example, the second player should recieve 25% of the stakes and the first player the other 75%. We then have a ratio of 3 to 1.

To solve this problem, Pascal came up with his arithmetic triangle and its binomial coefficient. But before giving out the answer to this problem[4], let's study Pascal's triangle in depth.

1.2 Construction

Let $\binom{n}{k}$ the binomial coefficient[5] which is equal to the value of the cell in the corresponding line (for n) and column (for k) as seen in figure 1.1.

Example (Binomial coefficients). $\binom{0}{0} = 1$, $\binom{1}{0} = 1$, $\binom{2}{1} = 2$, $\binom{4}{3} = 4$, $\binom{5}{2} = 10$

To construct Pascal's triangle means to find all the $\binom{n}{k}$ with n and k whole numbers and $0 \leq k \leq n$.

But to make some proofs easier, I am going to use an extension of it, where $\binom{n}{k}$ is defined for all $k \in \mathbb{Z}$ and for $k < 0$ or $k > n$ we have $\binom{n}{k} = 0$.

[1]Khayyam triangle in Iran and Yang Hui triangle in China
[2]Except in Italy where it is called Tartaglia's triangle
[3]But it was only published posthumously in 1654
[4]Answer page 10 for the impatients
[5]Read n on k

$n \backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Figure 1.1: Pascal's triangle

$\binom{n}{k}$ for $n < 0$ is usually not defined but some extensions exist.
Here are 3 common constructions of Pascal's triangle.

Construction 1 (Recursive construction)**.** $n \in \mathbb{N}$

$$\binom{0}{0} = 1 \tag{1.1}$$

$$\forall p \in \mathbb{Z}, p \neq 0 : \binom{0}{p} = 0 \tag{1.2}$$

$$\forall k \in \mathbb{Z} : \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \tag{1.3}$$

Remark. You can thus easily use a spreadsheet program (like Excel) to construct the triangle.

Construction 2 (In terms of n and k)**.** $(n, k) \in \mathbb{N}^2$

$$\forall k, 0 \leq k \leq n : \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.4}$$

Construction 3 (Binomial coefficients)**.** $n \in \mathbb{N}, (a, b) \in \mathbb{C}^2$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1.5}$$

Remark. The latest construction is known as the binomial theorem which alows you to quickly expand (if you know the binomial coefficients) something of the type : $(a + b)^n$ But it can be used in the other way around to find the binomial coefficients if you know the expansion of $(a + b)^n$.

Example.

$$(a+b)^1 = a + b \Rightarrow \binom{1}{0} = 1, \binom{1}{1} = 1 \tag{1.6}$$

$$(a+b)^2 = a^2 + 2ab + b^2 \Rightarrow \binom{2}{0} = 1, \binom{2}{1} = 2, \binom{2}{2} = 1 \tag{1.7}$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \Rightarrow \binom{3}{0} = 1, \binom{3}{1} = 3, \binom{3}{2} = 3, \binom{3}{3} = 1 \tag{1.8}$$

1.3 Proofs

We are going to prove that the three constructions are coherent. We are going to use the recursive construction in order to find the two others. Indeed, the way I define them, it is the construction with the biggest definition domain.

In order to find the two others constructions, we need first two lemmas which proofs are in the appendix A.1 on page 13.

Lemma 1. *Let* $n \in \mathbb{N}$*, then :* $\binom{n}{0} = 1$

Lemma 2. *Let* $n \in \mathbb{N}$*, then :* $\binom{n}{n} = 1$

1.3.1 Proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof by induction. We want to prove that if $(n,k) \in \mathbb{N}^2$ and $0 \leq k \leq n$ then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Base case. Let $n = 0$ and k such as $0 \leq k \leq n$. Then $k = 0$ and :

$$\frac{n!}{k!(n-k)!} = \frac{0!}{0!0!} = 1 \tag{1.9}$$

However $\binom{n}{k} = \binom{0}{0} = 1$. Thus for $n = 0$ and k such as $0 \leq k \leq n$ we have :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.10}$$

Inductive step. Let a $m \in \mathbb{N}$ and $p \in \mathbb{N}$ for which for every $p, 0 \leq p \leq m$, we have $\binom{m}{p} = \frac{m!}{p!(m-p)!}$.

If $p = 0$, then according to lemma 1 : $\binom{m+1}{p} = \binom{m+1}{0} = 1$. However $\frac{(m+1)!}{0!(m+1-0)!} = \frac{(m+1)!}{(m+1)!} = 1$. Thus for $p = 0$:

$$\binom{m+1}{p} = \frac{(m+1)!}{p!(m+1-p)!} \tag{1.11}$$

For p such as $0 \leq p < m$, we have $0 < p+1 \leq m$ and :

$$\binom{m+1}{p+1} = \binom{m}{p} + \binom{m}{p+1} = \frac{m!}{p!(m-p)!} + \frac{m!}{(p+1)!(m-(p+1))!} \tag{1.12}$$

$$= m!\left(\frac{1}{p!(m-p)!} + \frac{1}{(p+1)!(m-p-1)!}\right) \tag{1.13}$$

$$= m!\left(\frac{p+1}{(p+1)!(m-p)!} + \frac{m-p}{(p+1)!(m-p)!}\right) \tag{1.14}$$

$$= \frac{m! \times (m+1)}{(p+1)!(m-p+1-1)!} \tag{1.15}$$

$$= \frac{(m+1)!}{(p+1)! \times (m+1-(p+1))!} \tag{1.16}$$

If $p = m+1$, then according to lemma 2 : $\binom{m+1}{p} = \binom{m+1}{m+1} = 1$. However $\frac{(m+1)!}{(m+1)!(m+1-(m+1))!} = \frac{(m+1)!}{(m+1)!} = 1$. Thus for $p = m+1$:

$$\binom{m+1}{m+1} = \frac{(m+1)!}{(m+1)!(m+1-(m+1))!} \tag{1.17}$$

So for $0 \leq p \leq m+1$: $\binom{m+1}{p} = \frac{(m+1)!}{p!(m+1-p)!}$

Since the basis and inductive step have been performed, by mathematical induction for $(n,k) \in \mathbb{N}^2$ and k such as $0 \leq k \leq n$, we have :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.18}$$

So Pascal's triangle construction 1 and 2 are coherent. □

1.3.2 Proof of $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Proof by induction. Let $(a,b) \in \mathbb{C}^2$. We want to prove that if $n \in \mathbb{N}$ then $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Base case. The proof is obvious for $n = 0$, however, our inductive step needs the property to be initialized for at least $n = 1$. We then have :

$$\begin{cases} (a+b)^n = (a+b)^1 = a+b \\ \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^1 = a+b \end{cases} \Rightarrow (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1.19}$$

Inductive step. Let a $m \in \mathbb{N}$ for which $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$. Then :

$$\begin{aligned}
(a+b)^{m+1} &= (a+b)(a+b)^m && (1.20)\\
&= (a+b) \left(\sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k \right) && (1.21)\\
&= \left(\sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k \times (a+b) \right) && (1.22)\\
&= \left(\sum_{k=0}^{m} \binom{m}{k} \left(a^{m-k+1} b^k + a^{m-k} b^{k+1} \right) \right) && (1.23)\\
&= \left(\sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k \right) + \left(\sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^{k+1} \right) && (1.24)\\
&= \binom{m}{0} a^{m+1} + \left(\sum_{k=1}^{m} \binom{m}{k} a^{m+1-k} b^k \right) + \left(\sum_{k=0}^{m-1} \binom{m}{k} a^{m-k} b^{k+1} \right) + \binom{m}{m} b^{m+1} && (1.25)\\
&= \binom{m}{0} a^{m+1} + \left(\sum_{k=1}^{m} \binom{m}{k} a^{m+1-k} b^k \right) + \left(\sum_{k=1}^{m} \binom{m}{k-1} a^{m-(k-1)} b^{(k-1)+1} \right) + \binom{m}{m} b^{m+1} && (1.26)\\
&= \binom{m}{0} a^{m+1} + \left(\sum_{k=1}^{m} \binom{m}{k} a^{m+1-k} b^k + \binom{m}{k-1} a^{m-k+1} b^k \right) + \binom{m}{m} b^{m+1} && (1.27)\\
&= \binom{m}{0} a^{m+1} + \left(\sum_{k=1}^{m} a^{m+1-k} b^k \times \left[\binom{m}{k} + \binom{m}{k-1} \right] \right) + \binom{m}{m} b^{m+1} && (1.28)\\
&= \binom{m+1}{0} a^{m+1} + \left(\sum_{k=1}^{m} a^{m+1-k} b^k \times \binom{m+1}{k} \right) + \binom{m+1}{m+1} b^{m+1} && (1.29)\\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} a^{m+1-k} b^k && (1.30)
\end{aligned}$$

To go from line 1.27 to 1.28 we used lemmas 1 and 2 and the recursive construction.

Since the basis and inductive step have been performed, by mathematical induction, for $n \in \mathbb{N}$:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1.31}$$

So Pascal's triangle construction 1 and 3 are coherent. □

So Pascal's triangle recursive construction is coherent to the two others.

Chapter 2

Probability and other properties

2.1 Binomial trial

2.1.1 Definitions

Definition 1 (Bernoulli trial [5])**.** *In the theory of probability, a* Bernoulli trial *is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.*

Definition 2 (Binomial experiment [5])**.** *A* binomial experiment *is an experiment which consists of a fixed number n of statisticly independent Bernoulli trials, each with a propability of success p. A random variable corresponding to this experiment is denoted by $B(n,p)$ and is said to have a* binomial distribution*.*

2.1.2 Calculation

Let $(n,k) \in \mathbb{N}^2$, where $0 \leq k \leq n$ and $n \neq 0$. And let p such as $0 \leq p \leq 1$.

The probability $P(k)$ of exactly k successes in the binomial experiment $B(n,p)$, is :

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{2.1}$$

2.1.3 Proof

We want to prove the formula above.

Let a binomial distribution $B(n,p)$ where $n \in \mathbb{N}, n \neq 0$, and p such as $0 \leq p \leq 1$.

Let the random variable X such as $X \sim B(n,p)$, and $P(k) = P(X = k)$.

First part : $p^k \times (1-p)^{n-k}$

To work out $P(k)$ means to add the different probability of every single combination of successes and failure in which there are k successes.

But each combination has the same probability because it contains the same number of successes and failure and each trial are independent.

Each combination countains k successes, and thus $(n-k)$ failures (because there is a total of n trials). The probability of a success is p and of a failure is $(1-p)$ (because failure is the opposite evenement of success). Thus the probability of a particular combination is :

$$p^k \times (1-p)^{n-k} \tag{2.2}$$

Let $S(n,k)$ the number of combinations which countain k successes, then :

$$P(k) = S(n,k) \times p^k (1-p)^{n-k} \tag{2.3}$$

H	HHTT	All the combinations with 3 H and 2 T and beginning with H (in the box there is all the $S(4,2)$ combinations with 2 H and 2 T	All $S(5,3)$ combinations with 3 H and 2 T
H	HTHT		
H	HTTH		
H	THHT		
H	THTH		
H	TTHH		
T	HHHT	All the combinations with 3 H and 2 T and beginning with T (in the box there is all the $S(4,3)$ combinations with 3 H and 1 T	
T	HHTH		
T	HTHH		
T	THHH		

Thus : $S(5,3) = S(4,2) + S(4,3) = \binom{4}{2} + \binom{4}{3} = \binom{5}{3}$

Figure 2.1: All $S(5,3)$ combinations

Second part : $S(n,k) = \binom{n}{k}$

Let's now consider the different combinations. In these, H would represent a success, and T a failure. We want to work out the total number of different combinations we can do with k H and $(n-k)$ T.

Example. Let $B(4, \frac{1}{2})$, which modelize the experiment of flipping 4 times a coin. If we want 2 successes, we are looking for the number of combination with the same characters as HHTT.

There is a total of 6 different combinations : HHTT, HTHT, HTTH, THHT, THTH and TTHH. Thus $S(4,2) = 6$.

It appears obvious that with a times H and b times T or with b times H and a times T we can write the same number of combinations, because it is the number of each characters that matters and not the characters themselves. This leads us to the following property :

Property 1. *If* $(a,b) \in \mathbb{N}^2$

$$S(n,k) = S(n, n-k) \tag{2.4}$$

We also have the following lemma which is proved in the appendix A.2 on page 14.

Lemma 3. $n \in \mathbb{N}, n > 0$

$$S(n,n) = S(n,0) = 1 \tag{2.5}$$

Proof by induction. We want to prove that $S(n,k) = \binom{n}{k}$.

Base case. $S(n,k)$ is only defined[1] for $n > 0$. With lemma 3 we have : $S(1,0) = S(1,1) = 1$

But we know that $\binom{1}{0} = \binom{1}{1} = 1$, Thus :

$$\forall k \in \mathbb{N}, 0 \leq k \leq n : S(1,k) = \binom{1}{k} \tag{2.6}$$

Inductive step. Let's consider a $m \in \mathbb{N}$ for which for all $k \in \mathbb{N}, 0 \leq k \leq m$ we have $S(m,k) = \binom{m}{k}$.

The proof will proceed the same way as in figure 2.1.

Let $p \in \mathbb{N}$ where $0 < p \leq m$. The number of combinations we can write with p times T (failure) and $(m+1-p)$ times H (success) is equal to $S(m+1,p)$. We can divide all these combinations into 2 sets :

The ones beginning with H. If all these combinations begin with H, we are left with p T and $((m+1-p)-1)$ H to form other combinations. But with p T and $(m-p)$ H we can write $S(m,p)$ combinations. Thus the set we just consider is composed of $S(m,p)$ combinations.

The ones beginning with T. If we procceed using the same method, we now have $(p-1)$ T and $(m+1-p)$ H. We have thus a total of m characters. So with the characters, which are now left, we can write $S(m,p-1)$ combinations[2].

[1] If $n \leq 0$ then we have no characters to make combinations from.

[2] Or $S(m, m+1-p)$ combinations, but with property 1 : $S(m,p-1) = S(m,m+1-p)$

The two sets which we just consider contains all $S(m+1,p)$ combinations, thus :

$$S(m+1,p) = S(m,p-1) + S(m,p) = \binom{m}{p-1} + \binom{m}{p} = \binom{m+1}{p} \tag{2.7}$$

But to complete the inductive step we still need to check for $p=0$ and $p=m+1$.
With lemmas 1 and 2 we have : $\binom{m+1}{0} = \binom{m+1}{m+1} = 1$ and with lemma 3 we have $S(m+1,0) = S(m+1,m+1) = 1$. Thus :

$$S(m+1,0) = \binom{m+1}{0} \qquad S(m+1,m+1) = \binom{m+1}{m+1} \tag{2.8}$$

Thus for all $0 \leq k \leq m+1$ we have $S(m+1,k) = \binom{m+1}{k}$.

Since the basis and inductive step have been performed, by mathematical induction, for n a natural number and $k \in \mathbb{N}$:

$$0 \leq k \leq n, S(n,k) = \binom{n}{k} \tag{2.9}$$

Thus the probability of exactly k successes in the binomial experiment $B(n,p)$, is :

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{2.10}$$

□

We also proved at the same time an interesting property which can be deduced from property 1 and the equality $S(n,k) = \binom{n}{k}$:

Property 2. *$(n,k) \in \mathbb{N}^2, 0 \leq k \leq n$:*

$$\binom{n}{k} = \binom{n}{n-k} \tag{2.11}$$

Remark. This property is easier to prove using construction 2. Using this other method we can proved the property for $n = 0$.

2.2 Other properties

This section will contain some properties of Pascal's triangle and they will not be proven in this report. Many properties using the binomial coefficients exist [6], making it nearly impossible to do an exhaustive list of them. Thus, I just picked a few of them to present here.

Property 3. $\forall (n,k) \in \mathbb{N}^2, 0 \leq k \leq n : \binom{n}{k} \in \mathbb{N}$

Remark. This can be easily proven using the recursive construction.

Property 4 (Triangular numbers or sum of the n-th natural numbers)**.** *Let n a natural number :*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2} \tag{2.12}$$

Property 5 (Horizontal sums)**.** *Let n a natural number :*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{2.13}$$

Property 6. *Let n, k and h whole numbers :*

$$\binom{n}{h}\binom{n-h}{k} = \binom{n}{k}\binom{n-k}{h} \qquad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \tag{2.14}$$

Property 7 (Fibonacci number [7])**.** *Let $n \in \mathbb{N} - \{0\}$ and $F(n)$ denote the n^{th} Fibonacci number :*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = F(n+1) \tag{2.15}$$

2.3 The historical problem : Division of the stakes

With those new properties and the formula of the binomial distribution (equation 2.1), we can now answer the historical problem which saw Pascal's triangle born (at least in Pascal's mind).

I highly suggest you read again the paragraph about the problem on page 3.

The only criteria for the division of the stakes is how much rounds was needed by each player to reach the threshold when the game stopped. Indeed it is obvious that the division of the stakes should be the same if the score is 2/5 in a game of 8 rounds, or 14/17 in a game of 20 rounds (in both cases the required rounds to win is 6 for one player and 3 for the other).

So let's consider a game where one player requires α_1 rounds to win, and the other α_2. So hypothetically the maximum numbers of rounds which could have been played if the game had not been interrupted is $\alpha_1 + \alpha_2 - 1$.

At the end of all this rounds, there can only be one winner. Thus we can consider a situation where all the rounds are played, even though there is already a winner. We can then work out, using a binomial distribution, the odds of each player to win.

Let X_1 a random variable which counts the number of won rounds by the first player (from $\alpha_1+\alpha_2-1$ rounds), thus $X_1 \sim B(\alpha_1 + \alpha_2 - 1, 0.5)$. And for the second player we have : $X_2 \sim B(\alpha_1 + \alpha_2 - 1, 0.5)$.

We want the first player to have won at least α_1 rounds. We are thus looking for $P(X_1 \geq \alpha_1)$. We can then apply the same for the second player :

$$P(X_1 \geq \alpha_1) = \sum_{k=\alpha_1}^{\alpha_1+\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k} \times 0.5^k \times 0.5^{\alpha_1+\alpha_2-1-k} \tag{2.16}$$

$$= \sum_{k=\alpha_1}^{\alpha_1+\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k} \times 0.5^{\alpha_1+\alpha_2-1} \tag{2.17}$$

$$P(X_2 \geq \alpha_2) = \sum_{k=\alpha_2}^{\alpha_1+\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k} \times 0.5^{\alpha_1+\alpha_2-1} \tag{2.18}$$

We can simplify this and express it in the form of a ratio.

$P(X_1 \geq \alpha_1)$ and $P(X_2 \geq \alpha_2)$ are opposite evenements (if one player wins, the other loses), thus we have :

$$P(X_1 \geq \alpha_1) = 1 - P(X_2 \geq \alpha_2) \tag{2.19}$$

$$= P(X_2 < \alpha_2) \tag{2.20}$$

$$= P(X_2 \leq \alpha_2 - 1) \tag{2.21}$$

$$= \sum_{k=0}^{\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k} \times 0.5^{\alpha_1+\alpha_2-1} \tag{2.22}$$

So we have :

$$\frac{P(X_1 \geq \alpha_1)}{P(X_2 \geq \alpha_2)} = \frac{\sum_{k=0}^{\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k}}{\sum_{k=\alpha_2}^{\alpha_1+\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k}} \tag{2.23}$$

So the stakes should be divided between the two players with a ratio of :

$$\sum_{k=0}^{\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k} \text{ to } \sum_{k=\alpha_2}^{\alpha_1+\alpha_2-1} \binom{\alpha_1 + \alpha_2 - 1}{k} \tag{2.24}$$

Chapter 3

Fractals

3.1 Definition

Definition 3 (Fractals [8])**.** *Geometric fractals are fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole. Authorities disagree on the exact definition of fractals, but most usually elaborate on the basis ideas of self-similarity and an unusual relationship with the space a fractal is embedded in*[1]*.*

3.2 Sierpinski triangle

Sierpinski triangle is a fractal which is simple to create. It has the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles [9].

But it can also be created using Pascal's triangle by coloring all the odd numbers as shown in figure 3.1.

You can thus easily write an algorithm to create a Sierpinski triangle using Pascal's triangle and its binomial coefficients. Such an algorithm would check the parity of all $\binom{n}{k}$ and color them in consequence. The one I wrote can be find in appendix B on page 15 and the result is the figure 3.2.

You can also prove that with this construction Sierpinski triangle is indeed a fractal. I wrote a proof which can be found at : `http://bit.ly/sierpinski2014`

Many other fractals exist, which are sometimes as simple to create. For example :

- Sierpinski carpet
- Menger sponge (the 3-dimensional version of Sierpinski carpet)
- Koch snowflake

These three fractals are very basic ones, and if you want to know more about those figures you should also check the Mandelbrot set.

Conclusion. This ends out journey which brought us from combinatorics and the division of stakes to curious mathematical objects known as fractals.

[1]Some 1-dimension fractals look like surfaces (a 2-dimension shape)

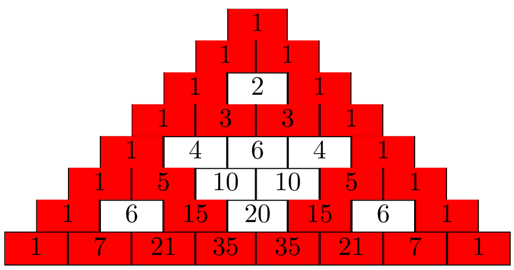

Figure 3.1: The use of Pascal's triangle for the creation of a Sierpinski triangle

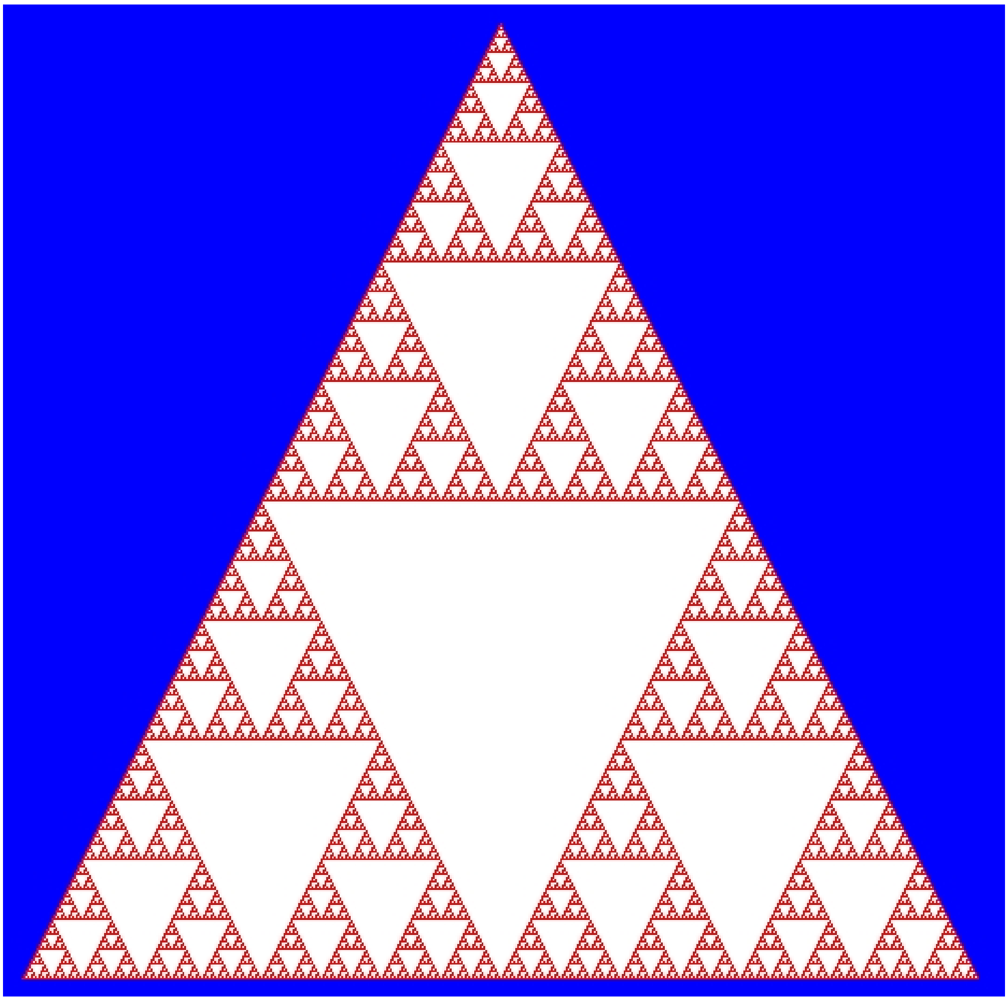

Figure 3.2: A Sierpinski triangle

Appendix A

Complementary proofs

A.1 Proofs of lemmas 1 and 2

To prove this lemmas we are going to use the recursive construction[1] on page 4.

Proof of lemma 1 on page 4 : $n \in \mathbb{N} \Rightarrow \binom{n}{0} = 1$. Let $n \in \mathbb{N}$.

Base case. We know that $\binom{0}{0} = 1$ and that $\forall p \in \mathbb{Z}, p \neq 0 : \binom{0}{p} = 0$, thus $\forall p < 0 : \binom{0}{p} = 0$.

Inductive step. Let a $m \in \mathbb{N}$ for which $\binom{m}{0} = 1$ and $\forall d \in \mathbb{Z}, d < 0 : \binom{m}{d} = 0$. Thus :

$$\binom{m+1}{0} = \binom{m}{-1} + \binom{m}{0} = 0 + 1 = 1 \tag{A.1}$$

And $\forall d \in \mathbb{Z}, d < 0 : \binom{m+1}{d} = \binom{m}{d-1} + \binom{m}{d} = 0 + 0 = 0$

Since the basis and inductive step have been performed, by mathematical induction, for $n \in \mathbb{N}$:

$$\binom{n}{0} = 1 \tag{A.2}$$

□

Proof of lemma 2 on page 4 : $n \in \mathbb{N} \Rightarrow \binom{n}{n} = 1$. Let $n \in \mathbb{N}$.

Base case. We know that $\binom{0}{0} = 1$ and that $\forall p \in \mathbb{Z}, p \neq 0 : \binom{0}{p} = 0$, thus $\forall p > 0 : \binom{0}{p} = 0$.

Inductive step. Let a $m \in \mathbb{N}$ for which $\binom{m}{m} = 1$ and $\forall d \in \mathbb{Z}, d > m : \binom{m}{d} = 0$. Thus :

$$\binom{m+1}{m+1} = \binom{m}{m} + \binom{m}{m+1} = 1 + 0 = 1 \tag{A.3}$$

And $\forall d \in \mathbb{Z}, d > m + 1 : d - 1 > m$ and $d > m$, thus $\binom{m+1}{d} = \binom{m}{d-1} + \binom{m}{d} = 0 + 0 = 0$

Since the basis and inductive step have been performed, by mathematical induction, for $n \in \mathbb{N}$:

$$\binom{n}{n} = 1 \tag{A.4}$$

□

[1] Construction 1

A.2 Proof of lemma 3 on page 8

Let $S(n, k)$, $(n, k) \in \mathbb{N}^2, n > 0$, the number of combinations which can be formed using k H and $(n - k)$ T.

We want to prove that $S(n, n) = S(n, 0) = 1$

Proof. A combination with only one type of characters is unique.

If $k = 0$, then there is 0 H and n T. Thus we would only have some T to form a combination and thus there would be a unique one.

If $k = n$, then there is n H and 0 T. Thus we would only have some H to form a combination and thus there would be a unique one.

Thus for n a natural number :

$$S(n, n) = S(n, 0) = 1 \tag{A.5}$$

□

Appendix B

Algorithm of the Sierpinski triangle

Algorithm 1: Algorithm of the Sierpinski triangle

```
Data: The width width of the screen in pixel and the height height of the screen in pixel
Output: Displays Sierpinski triangle on the screen

Initialize i and j as integers;
Set the screen's background to blue;

for i ← 0 to height do
    for j ← 0 to i do
        select pixel ((width+i)/2 − j, i) ;      /* pixel on column (width+i)/2 − j and line i */
        if isEven(i, j) then    /* check if (i choose j) is even (for isEven see algorithm 2) */
            set current pixel to white;
        else
            set current pixel to red;
```

Algorithm 2: isEven function

```
Function isEven(n, k)                    /* check if a binomial coefficient is even */
    Initialize count, temp1 and temp2 as integers;

    count ← 0 ;          /* count counts the number of time (n choose k) can be divided by 2 */
    if n ≥ 2 then
        for temp1 ← k + 1 to n do   /* counts how many times n!/k! can be divided by 2 */
            temp2 ← temp1;
            while (temp2 ≡ 0 (mod 2)) and (temp2 ≠ 0) do
                temp2 ← temp2/2;
                count ← count + 1 ;                      /* store this number in count */

        for temp1 ← 2 to n − k do                   /* repeat the process for (n − k)! */
            temp2 ← temp1;
            while (temp2 ≡ 0 (mod 2)) and (temp2 ≠ 0) do
                temp2 ← temp2/2;
                count ← count − 1 ;               /* decrement count by this ammount */

    if count > 0 then
        return true;
    else
        return false;
```

References

[1] Clifford A. Pickover. *The Math Book*. Sterlling Publishing, 2009. p. 146.

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[8] Fractal. en.wikipedia.org/wiki/Fractal.

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