Mechanisation of the AKS Algorithm

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Outline

Mechanisation of AKS Algorithm – Road Map

- Foundation Work:
  - Build Monoid theory in HOL4.($\surd$)
  - Build Group theory from Monoid theory.($\surd$)
  - Build Ring theory using Group and Monoid.($\surd$)
  - Build Field theory using Ring and Group.($\surd$)
  - Build Polynomial theory using Field and Ring.($\surd$)
- Apply to AKS (to be amended):
  - Code in HOL4: *AKS n* that returns true or false upon input n.
  - Prove in HOL4: *AKS n* returns true iff n is prime.
  - Prove in HOL4: number of steps of *AKS n* is bound by $O(\log^k n)$.
  -

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- Apply to AKS (now):
  - Define: a number n to be *AKS n* if n satisfies certain properties.
  - Prove (the AKS main theorem): *AKS n* is true iff n is prime.
  - Code: an algorithm to verify that a given number n is *AKS n*.
  - Prove: number of steps of the algorithm is bounded by $O(\log^k n)$.

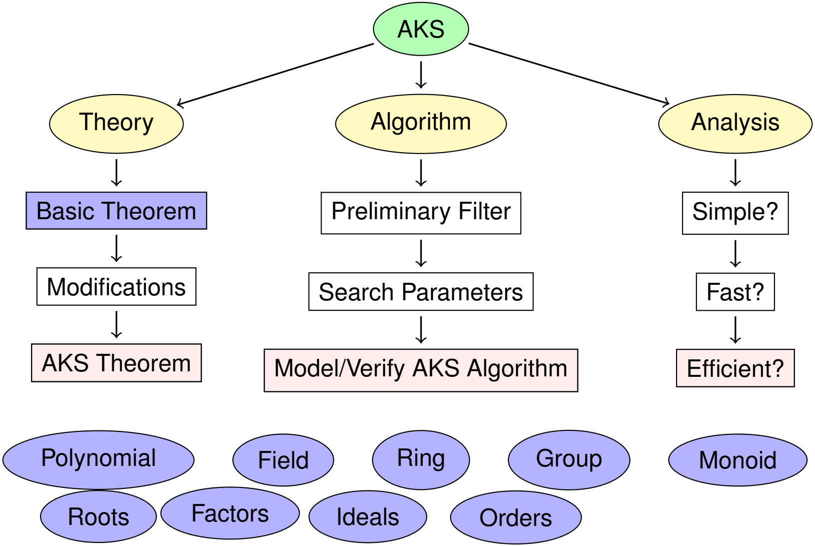
Work Summary — from Elementary

- What Have I Done?
  - Consolidation of basics: Monoid, Group, Ring, Field.
  - Expanded theory of polynomials over Ring and Field.
  - Developed theory of orders for elements in Monoid and Group.
  - Developed polynomial divisions with Ring and Field coefficients.

Work Summary — to Advanced

- What Have I Done?
  - Developed theory of ring ideals and quotient rings.
  - Proved that quotient ring by a maximal ideal is a Field.
  - Developed theory of irreducible elements in a Ring.
  - Developed theory of Principal Ideal Rings (PIR).
  - Proved in PIR, the principal ideal of an irreducible element is maximal.
  - Proved that Euclidean Rings are Principal Ideal Rings.
  - Proved that division property in a polynomial ring makes it Euclidean.

Current Work



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HOL session:

```
- g `1 + 1 = 2`;
- e (decide_tac);
> val it =
    Initial goal proved.
    |- 1 + 1 = 2 :  proof
```

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HOL Tutorial's proof:

- Define: $\texttt{FACT}\ n = \prod_{m=1}^{n} m$.
- Prove properties of $\texttt{FACT}\ n$, *e.g.* $\forall n\, m.\ 1 \leq m \leq n \Rightarrow m \mid \texttt{FACT}\ n$.
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HOL builds up a tool: the `Factorial` library.

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- Consider the cosets of subgroup H.
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- Prove that for finite group G, $|G/H| = \dfrac{|G|}{|H|}$.
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High-level libraries have more general applications.

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How to program (OOP style):

- Design the objects.
- Develop a hierarchy of objects.
- Put features into libraries.
- Adopt a modular approach and make use of libraries.

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- From sets with elements of type α, we can define various algebraic structures: $\alpha\,\texttt{group}$, $\alpha\,\texttt{ring}$, $\alpha\,\texttt{field}$.
- Polynomials over an $\alpha\,\texttt{field}$ are simply $\alpha\,\texttt{poly} = \alpha\,\texttt{list}$.
- These polynomials themselves form a Ring, an $\alpha\,\texttt{poly}\,\texttt{ring}$.
- Theorems proved for $\alpha\,\texttt{ring}$ can be lifted to $\alpha\,\texttt{poly}\,\texttt{ring}$.

Possible Timeline

Revised thesis plan:

June, 2015:	AKS Theorem
June, 2016:	Model/Verify AKS Algorithm
June, 2017:	Complexity/Efficiency
December, 2017:	Thesis written (hopefully!)

My official start-date was 7 April 2012

Switched to part-time to extend original deadline (7 March 2016).