# Mechanisation of the AKS Algorithm

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# Outline

1. Introduction
   - Road Map
   - Current Plan
   - Deterministic Test
2. Deterministic Primality Testing
   - Basic Idea
   - Basic Theorem
   - AKS Theorem

# Mechanisation of AKS Algorithm – Road Map

- Foundation Work:
  - Build Monoid theory in HOL4.
  - Build Group theory from Monoid theory.
  - Build Ring theory using Group and Monoid.
  - Build Field theory using Ring and Group.
  - Build Polynomial theory using Field and Ring.
- Apply to AKS:
  - Code in HOL4: *AKS n* that returns true or false upon input $n$.
  - Prove in HOL4: *AKS n* returns true iff $n$ is prime.
  - Prove in HOL4: number of steps of *AKS n* is bound by $O(\log^k n)$.

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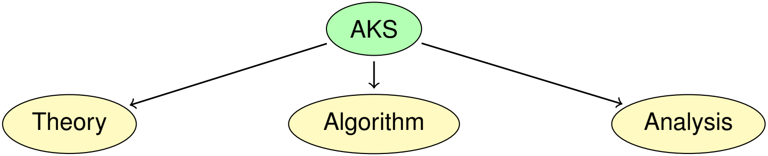

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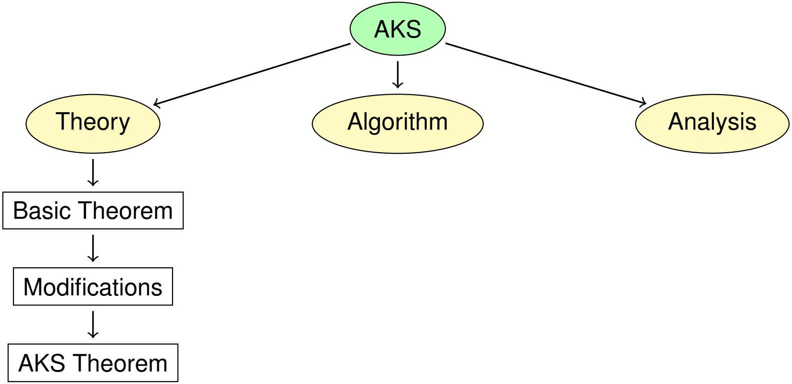

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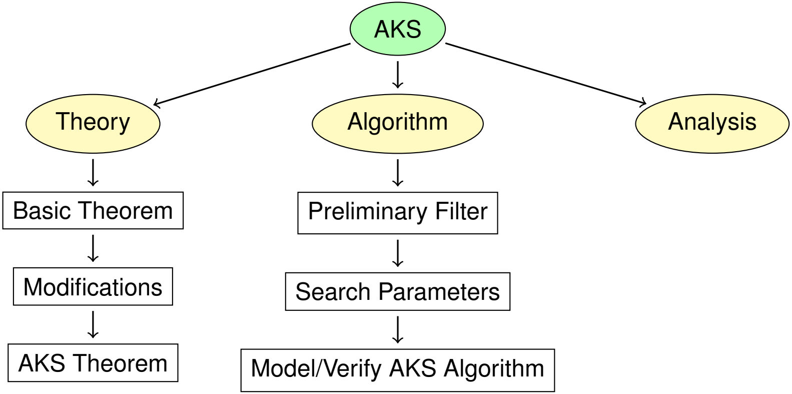

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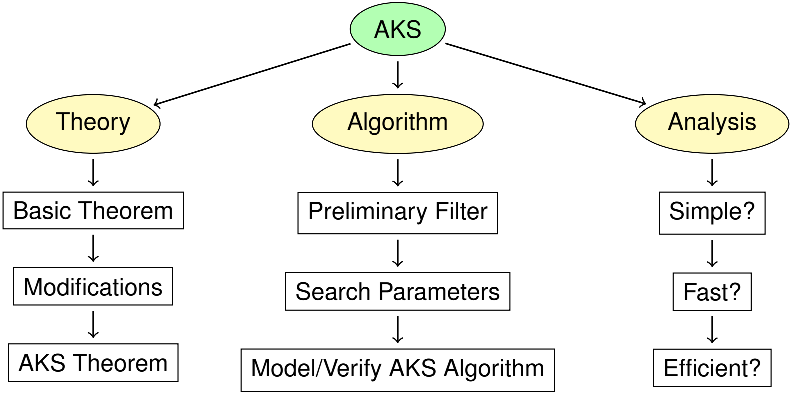

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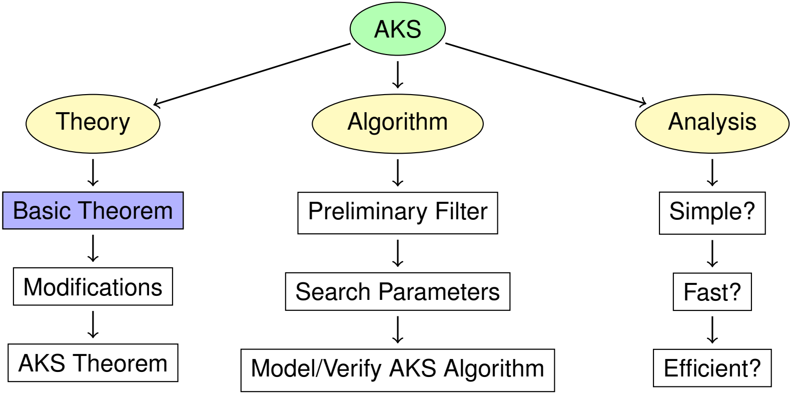

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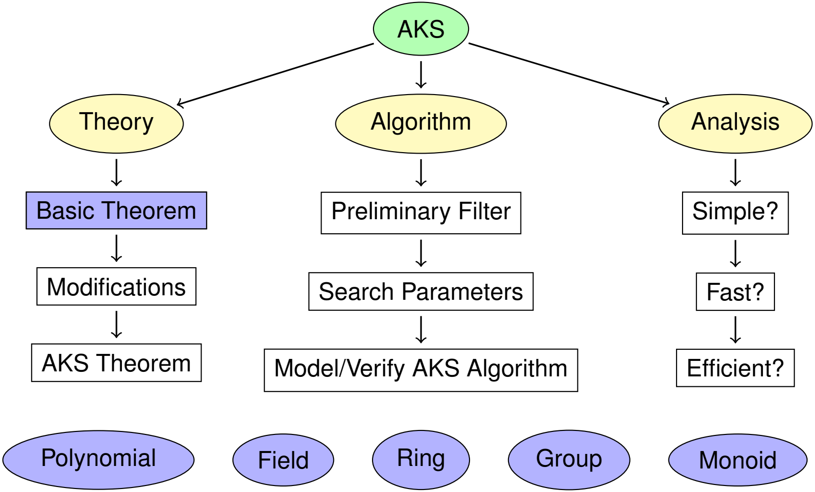

# Primality Testing

Is $n$ prime?

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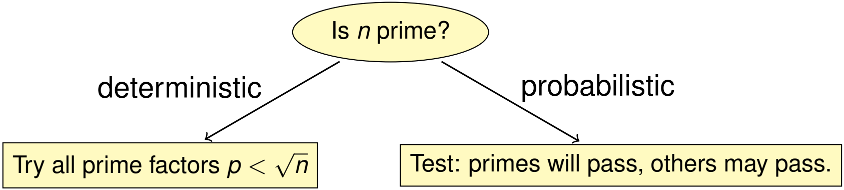

# Primality Testing

Is $n$ prime?

deterministic

Try all prime factors $p < \sqrt{n}$

$\downarrow$

Try 1st prime $\longrightarrow$ no

$\downarrow$

probabilistic

Test: primes will pass, others may pass.

$\downarrow$

Try one test $\longrightarrow$ no

$\downarrow$ 80% sure

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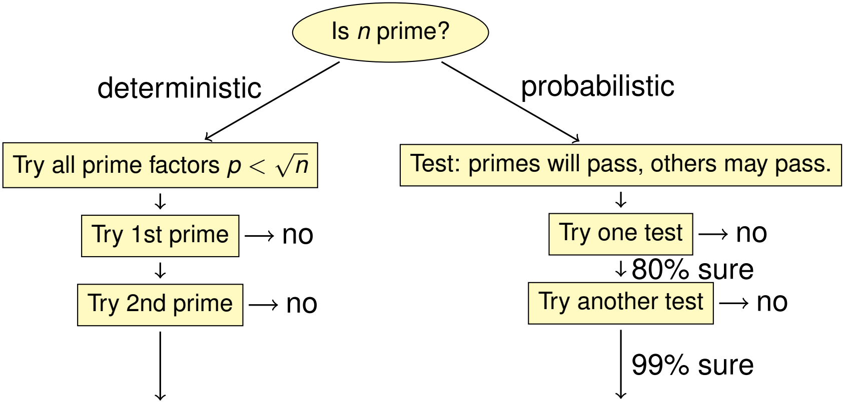

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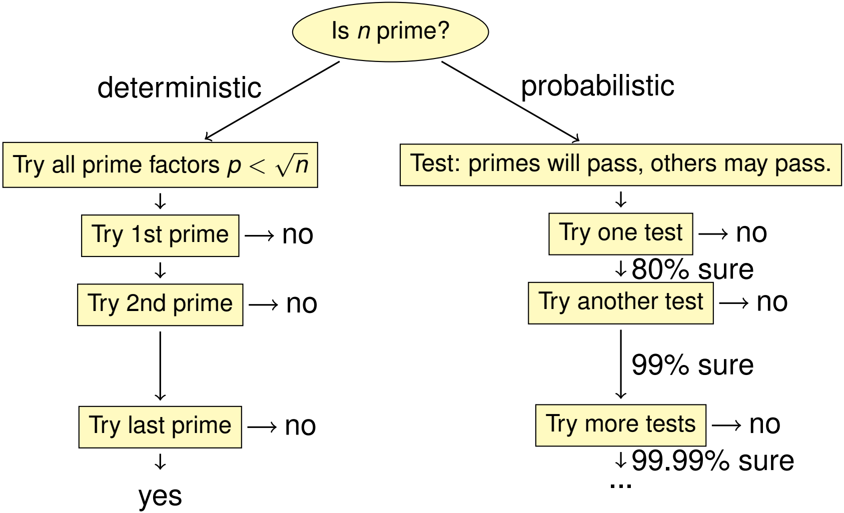

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- Prime $n \iff n > 1$ and $n$ divides all its non-unit Binomials.

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| $(x+1)^2$ | 1 | 2 | 1 | | | | | |
| $(x+1)^3$ | 1 | 3 | 3 | 1 | | | | |
| $(x+1)^4$ | 1 | 4 | 6 | 4 | 1 | | | |
| $(x+1)^5$ | 1 | 5 | 10 | 10 | 5 | 1 | | |
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- Famous Pascal's triangle, with binomial coefficients $\binom{n}{k}$.

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- Solution: $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$.

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- Theorem: prime $n \Leftrightarrow n > 1$ and $(x+1)^n \equiv x^n + 1 \pmod{n}$.

# Polynomial Identity for Primes

**Theorem**

*Given a number $n$, let $c$ be coprime with $n$; i.e. $gcd(c, n) = 1$.*
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- By coprime with $c$, $n$ cannot divide $c^{(n-k)}$, *i.e.* $c^{(n-k)} \not\equiv 0 \mod n$.
- Therefore, $\binom{n}{k} \equiv 0 \mod n$, for $0 < k < n$, or prime $n$ by "key". $\square$

# Towards AKS

Basic Theorem:
prime $n \iff n > 1$ and $(x + c)^n \equiv x^n + c \mod n$ for any coprime $c$.

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## Theorem

*Theorem is broken:*
*Assume* $n > 1$,
*prime* $n \Rightarrow (x+c)^n \equiv x^n + c \mod (n, x^r - 1)$ *for coprimes:* $1 \le c \le s$.
*prime* $n \not\Leftarrow (x+c)^n \equiv x^n + c \mod (n, x^r - 1)$ *for coprimes:* $1 \le c \le s$.
*(polynomial remainders of degree up to* $r$ *have less number of terms!)*

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## Theorem

*Theorem found by AKS team:*
*Given* $n > 1$,
*there exists suitable parameters* $r$ *and* $s$ *related to* $n$*, such that:*
$(x + c)^n \equiv x^n + c \mod (n, x^r - 1)$ *for coprimes:* $1 \le c \le s$
$\Rightarrow$ *$n$ is a prime power.*

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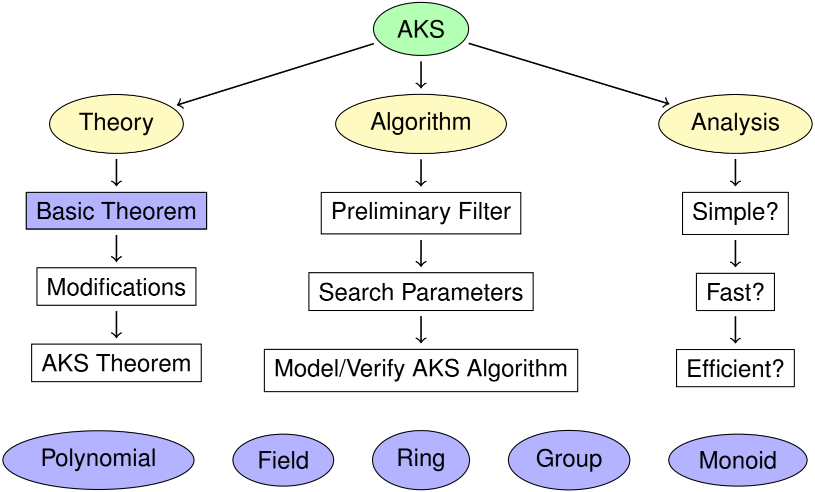
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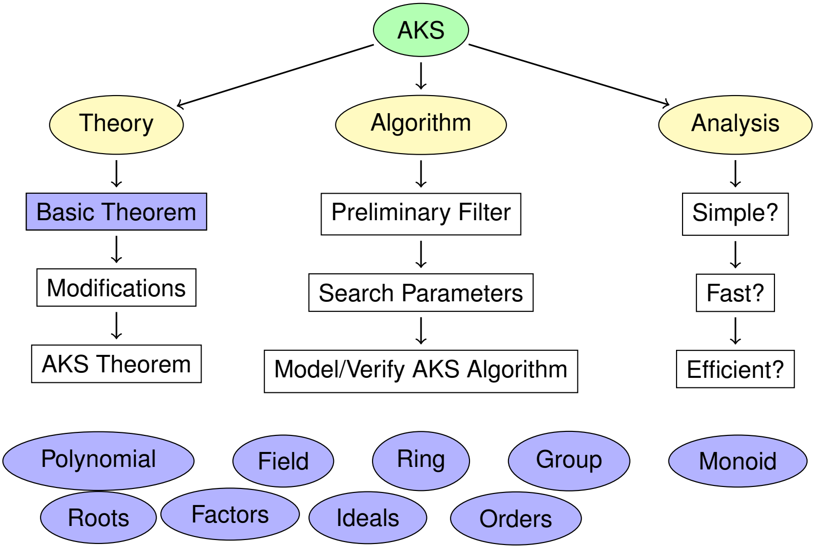
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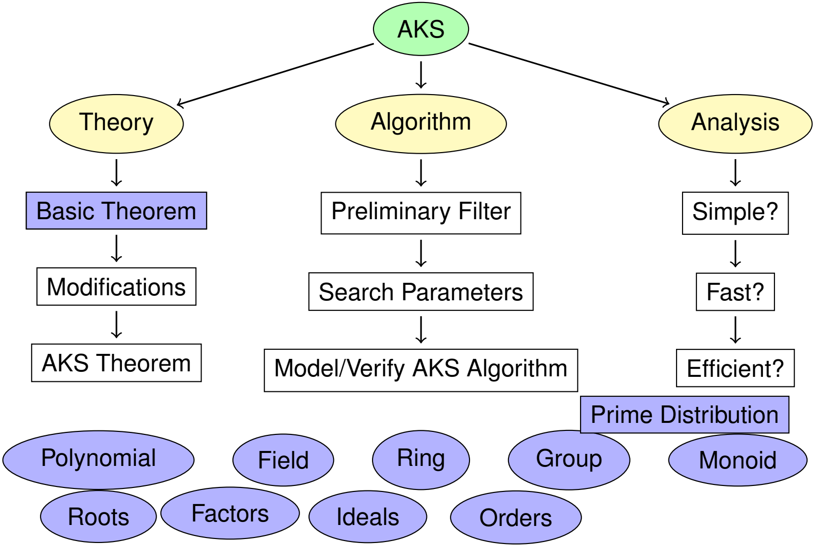
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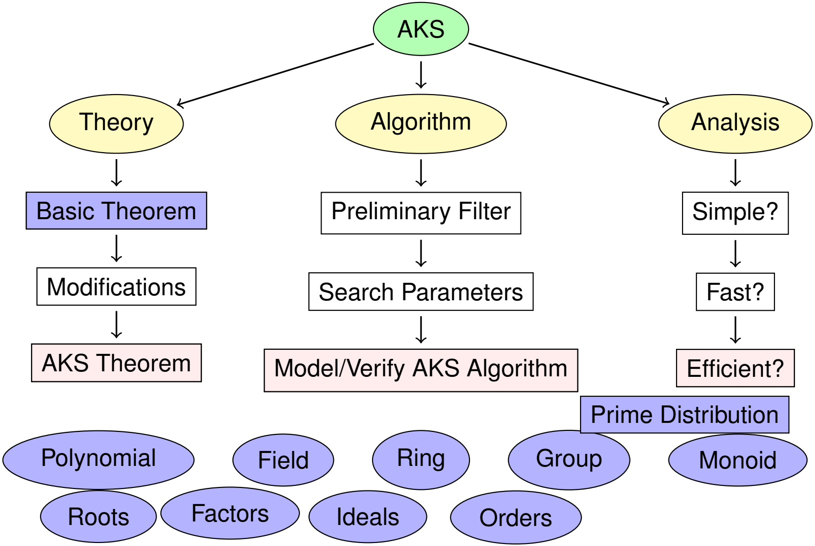
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Thesis plan:

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| **(end of) 2014:** | AKS Theorem |
| **June, 2015:** | Model/Verify AKS Algorithm |
| **June, 2016:** | Complexity/Efficiency |
| **December, 2016:** | Thesis written(!) |

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My official start-date was 7 April 2012

My latest possible submission date is $\sim$4 years later: 7 March 2016

If necessary, will switch to part-time to extend this deadline

# The Key - Part 1

## Theorem

*Prime n* $\Leftrightarrow n > 1$ *and n divides* $\binom{n}{k}$ *for* $0 < k < n$.

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- Recall binomial formula: $k!\binom{n}{k} = n(n-1)(n-2)...(n-k+1)$
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- Surely, $p$ divides RHS. Thus $p$ also divides LHS.
- Since all $k < p$, prime $p$ cannot divide $k!$.
- Therefore, $p$ must divide $\binom{p}{k}$. $\square$

# The Key - Part 2

**Theorem**

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- Assume $n$ is not prime, then it has a prime factor $p$ and $p < n$.
- Let $k = p$, divide by $n$: $\quad p!\frac{\binom{n}{p}}{n} = (n-1)(n-2)...(n-p+1)$

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- Note that $\frac{\binom{n}{p}}{n}$ is an integer, since $n$ divides all non-unit binomials.
- Therefore $p$ divides LHS. So $p$ must also divide RHS.

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- A contradiction — $n$ must be prime! $\square$