Proof Pearl: Bounding Least Common Multiples with Triangles

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Abstract. We present a proof of the fact that $2^n \leq \mathrm{lcm}\{1, 2, 3, \ldots, (n + 1)\}$. This result has a standard proof *via* an integral, but our proof is purely number theoretic, requiring little more than list inductions. The proof is based on manipulations of a variant of Leibniz's Harmonic Triangle, itself a relative of Pascal's better-known Triangle.

1 Introduction

The least common multiple of the consecutive natural numbers has a lower bound[1]:

$$2^n \leq \mathrm{lcm}\{1, 2, 3, \ldots, (n + 1)\}$$

This result is a minor (though important) part of the proof of the complexity of the "PRIMES is in P" AKS algorithm (see below for more motivational detail). A short proof is given by Nair [10], based on a sum expressed as an integral. That paper ends with these words:

> *It also seems worthwhile to point out that there are different ways to prove the identity implied [...], for example, [...] by using the difference operator.*

Nair's remark indicates the possibility of an elementary proof of the above number-theoretic result. Nair's integral turns out to be an expression of the beta-function, and there is a little-known relationship between the beta-function and Leibniz's harmonic triangle [2]. The harmonic triangle can be described as the difference table of the harmonic sequence: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ (*e.g.*, as presented in [3]).

Exploring this connection, we work out an interesting proof of this result that is both clear and elegant. Although the idea has been sketched in various sources (*e.g.*, [9]), we put the necessary pieces together in a coherent argument, and prove it formally in HOL4.

Overview We find that the rows of denominators in Leibniz's harmonic triangle provide a trick to enable an estimation of the lower bound of least common multiple (LCM) of consecutive numbers. The route from this row property to the LCM bound is subtle: we exploit an LCM property of triplets of neighboring elements in the denominator triangle. We shall show how this property gives a wonderful proof of the LCM bound for consecutive numbers in HOL4:

Theorem 1. *Lower bound for LCM of consecutive numbers.*

$\vdash 2^n \leq$ list_lcm $[1 \;..\; n + 1]$

* NICTA is funded by the Australian Government through the Department of Communications and the Australian Research Council through the ICT Centre of Excellence Program.

[1] We use $(n + 1)$ here since we allow $n = 0$.

where list_lcm *is the obvious extension of the binary* lcm *operator to a list of numeric arguments. This satisfies, for example, the following properties:*

$\vdash$ list_lcm $(h::t)$ = lcm h (list_lcm t)
$\vdash$ list_lcm $(l_1 \frown l_2)$ = lcm (list_lcm l_1) (list_lcm l_2)
$\vdash$ list_lcm (REVERSE ℓ) = list_lcm ℓ

Motivation This work was initiated as part of our mechanization of the AKS algorithm [1], the first unconditionally deterministic polynomial-time algorithm for primality testing. As part of its initial action, the algorithm searches for a parameter k satisfying a condition dependent on the input number. The major part of the AKS algorithm then involves a for-loop whose count depends on the size of k.

In our first paper on the correctness (but not complexity) of the AKS algorithm [4], we proved the existence of such a parameter k on general grounds, but did not give a bound. Now wanting to also show the complexity result for the AKS algorithm, we must provide a tight bound on k. As indicated in the AKS paper [1, Lemma 3.1], the necessary bound can be derived from a lower bound on the LCM of consecutive numbers.

Historical Notes Pascal's arithmetic triangle (c1654) is well-known, but Leibniz's harmonic triangle (1672) has been comparatively neglected. As reported by Massa Esteve and Delshams [5], Pietro Mengoli investigated certain sums of special form in 1659, using a combinatorial triangle identical to the harmonic triangle. Those same sums are the basis of Euler's beta-function (1730) defined by an integral.

In another vein, Hardy and Wright's *Theory of Numbers* [7] related the LCM bound of consecutive numbers to the Prime Number Theorem, which work was followed up by Nair [10], giving the bound in Theorem 1 through application of the beta-function.

Our approach to prove Theorem 1 is inspired by Farhi [6], in which a binomial coefficient identity, equivalent to our Theorem 6, was established using Kummer's theorem. A direct computation to relate both results of Nair and Farhi was given by Hong [8].

Paper Structure The rest of this paper is devoted to explaining the mechanised proof of this result. We give some background to Pascal's and Leibniz's triangles in Section 2. Section 3 discusses two forms of the Leibniz's triangle: the harmonic form and the denominator form, and proves the important LCM property for our Leibniz triplets. Section 4 shows how paths in the denominator triangle can make use of an LCM exchange property, eventually proving that both the consecutive numbers and a row of the denominator triangle share the same LCM. In Section 5, we apply this LCM relationship to give a proof of Theorem 1, and conclude in Section 6.

HOL4 Notation All statements starting with a turnstile ($\vdash$) are HOL4 theorems, automatically pretty-printed to LaTeX from the relevant theory in the HOL4 development. Generally, our notation allows an appealing combination of quantifiers ($\forall$, $\exists$), logical connectives ($\wedge$ for "and", $\Rightarrow$ for "implies", and $\iff$ for "if and only if"). Lists are enclosed in square-brackets [ ], with members separated by semicolon (;), using infix operators :: for "cons", $\frown$ for append, and . . for inclusive range. Common list operators are: LENGTH, SUM, REVERSE, MEM for list member, and others to be introduced as required. Given a binary relation $\mathcal{R}$, its reflexive and transitive closure is denoted by $\mathcal{R}^*$.

HOL4 Sources Our proof scripts, one for the Binomial Theory and one for the Triangle Theory, can be found at `https://bitbucket.org/jhlchan/hol/src/`, in the sub-folder `algebra/lib`.

2 Background

2.1 LCM Lower Bound for a List

The following observation is simple:

Theorem 2. *The least common multiple of a list of positive numbers equals at least its average.*

$\vdash$ ($\forall x$. MEM x ℓ $\Rightarrow$ $0 < x$) $\Rightarrow$ SUM ℓ $\leq$ LENGTH ℓ $\times$ list_lcm ℓ

Proof. For a list ℓ, since every element is nonzero, list_lcm ℓ is also nonzero. There are LENGTH ℓ elements, and each element $x \leq$ list_lcm ℓ. Therefore adding together LENGTH ℓ copies of list_lcm ℓ cannot be smaller than their sum, which is SUM ℓ. □

A naïve application of this theorem to the list of consecutive numbers gives a trivial and disappointing LCM lower bound. For an ingenious application of the theorem to obtain the better LCM lower bound in Theorem 1, we turn to Leibniz's Triangles, close relatives of Pascal's Triangle.

2.2 Pascal's Triangle

Pascal's well-known triangle (first in Figure 1) can be constructed as follows:

- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
\qquad
\begin{array}{ccccccccccccc}
 & & & & & & \frac{1}{1} & & & & & & \\
 & & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
 & & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & & \\
 & & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} & & & \\
 & & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & & \\
 & \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} & \\
\frac{1}{7} & & \frac{1}{42} & & \frac{1}{105} & & \frac{1}{140} & & \frac{1}{105} & & \frac{1}{42} & & \frac{1}{7}
\end{array}
$$

Fig. 1. Pascal's and Leibniz's Triangles

The entries of Pascal's triangle (the k-th element on n-th row) are binomial coefficients $\binom{n}{k}$, with the n-th row sum: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Since Leibniz's triangle (see Section 2.3 below) will be defined using Pascal's triangle, we include the binomials as a foundation in our HOL4 implementation, proving the above result:

Theorem 3. *Sum of a row in Pascal's Triangle.*

$\vdash$ SUM ($\mathcal{P}_{\text{row}}$ n) = 2^n

We use ($\mathcal{P}_{\text{row}}$ n) to represent the n-th row of the Pascal's triangle, counting n from 0.

2.3 Leibniz's Harmonic Triangle

Leibniz's harmonic triangle (second in Figure 1) can be similarly constructed:

- Each boundary entry: $\dfrac{1}{(n+1)}$ for the n-th row, with n starting from 0.
- Each entry (inside or not): sum of two immediate children.

With the boundary entries forming the harmonic sequence, this Leibniz's triangle is closely related to Pascal's triangle. Denoting the harmonic triangle entries (also the k-th element on n-th row) by $\begin{bmatrix} n \\ k \end{bmatrix}$, then it is not hard to show (*e.g.*, [2]) from the construction rules that:

- $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{1}{(n+1)\binom{n}{k}}$
- $\displaystyle\sum_{k=0}^{n} \binom{n}{k} \begin{bmatrix} n \\ k \end{bmatrix} = 1$

Therefore all entries of the harmonic triangle are unit fractions. So, we choose to work with Leibniz's "Denominator Triangle", by picking only the denominators of the entries. This allows us to deal with whole numbers rather than rational numbers in HOL4.

3 Leibniz's Denominator Triangle and Its Triplets

Taking the denominators of each entry of Leibniz's Harmonic Triangle to form Leibniz's Denominator Triangle, denoted by $\mathcal{L}$, we define its entries in HOL4 *via* the binomial coefficients:

Definition 1. *Denominator form of Leibniz's triangle: k-th entry at n-th row.*

$$\vdash\ \mathcal{L}\ n\ k = (n + 1) \times \binom{n}{k}$$

row n \ column k	$k=0$,	$k=1$,	$k=2$,	$k=3$,	$k=4$,	$k=5$,	$k=6$, $\cdots$
$n=0$	1						
$n=1$	2	2					
$n=2$	3	6	3				
$n=3$	4	12	12	4			
$n=4$	5	20	30	20	5		
$n=5$	6	30	60	60	30	6	
$n=6$	7	42	105	140	105	42	7

Table 1. Leibniz's Denominator Triangle. A typical triplet is marked.

The first few rows of the denominator triangle are shown (Table 1) in a vertical-horizontal format. Evidently from Definition 1, the n-th horizontal row is just a multiple of the n-th row in Pascal's triangle by a factor $(n+1)$, and the left vertical boundary consists of consecutive numbers:

$$\vdash\ \mathcal{L}\ n\ 0 = n + 1$$

Within this vertical-horizontal format, we identify L-shaped “Leibniz triplets” rooted at row n and column k, involving three entries:

- the top of the triplet being α_{nk}, and
- its two child entries as β_{nk} and γ_{nk} on the next row.

	Denominator Triangle	Harmonic Triangle
	$\cdots$	$\cdots$
row $\cdots$	$\cdots\ \cdots$	$\cdots$
row n	$\cdots\ \alpha_{nk}\ \cdots$	$\cdots\ \frac{1}{\alpha_{nk}}\ \cdots$
row $(n+1)$	$\cdots\ \beta_{nk}\ \gamma_{nk}\ \cdots$	$\cdots\ \frac{1}{\beta_{nk}}\ \cdots\ \frac{1}{\gamma_{nk}}\ \cdots$
row $\cdots$	$\cdots\ \cdots\ \cdots\ \cdots\ \cdots$	$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$
	$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$	$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$

Table 2. The Leibniz triplet

In other words, we can define the constituents of a typical Leibniz triplet as:

$\vdash\ \alpha_{nk}\ =\ \mathcal{L}\ \ n\ \ k$

$\vdash\ \beta_{nk}\ =\ \mathcal{L}\ \ (n\ +\ 1)\ \ k$ $\qquad \vdash\ \gamma_{nk}\ =\ \mathcal{L}\ \ (n\ +\ 1)\ \ (k\ +\ 1)$

Note that the values α_{nk}, β_{nk} and γ_{nk} occur as denominators in Leibniz’s original harmonic triangle, corresponding to the situation that the entry $\dfrac{1}{\alpha_{nk}}$ has immediate children $\dfrac{1}{\beta_{nk}}$ and $\dfrac{1}{\gamma_{nk}}$ (refer to Table 2). By the construction rule of harmonic triangle, we should have:

$$\frac{1}{\alpha_{nk}} = \frac{1}{\beta_{nk}} + \frac{1}{\gamma_{nk}}, \quad \text{or} \quad \frac{1}{\gamma_{nk}} = \frac{1}{\alpha_{nk}} - \frac{1}{\beta_{nk}}$$

which, upon clearing fractions, becomes:

$$\alpha_{nk}\ \times\ \beta_{nk}\ =\ \gamma_{nk}\ \times\ (\beta_{nk}\ -\ \alpha_{nk})$$

Indeed, it is straightforward to show that our definition of $(\mathcal{L}\ \ n\ \ k)$ satisfies this property:

Theorem 4. *Property of a Leibniz triple in Denominator Triangle.*

$\vdash\ \alpha_{nk}\ \times\ \beta_{nk}\ =\ \gamma_{nk}\ \times\ (\beta_{nk}\ -\ \alpha_{nk})$

This identity for a Leibniz triplet is useful for computing the entry γ_{nk} from previously calculated entries α_{nk} and β_{nk}. Indeed, the entire Denominator Triangle can be constructed directly out of such overlapping triplets:

- Each left boundary entry: $(n+1)$ for the n-th row, with n starting from 0.
- Each Leibniz triplet: $\gamma_{nk} = \dfrac{\alpha_{nk}\ \times\ \beta_{nk}}{\beta_{nk}\ -\ \alpha_{nk}}$.

This is also the key for the next important property of the triplet.

3.1 LCM Exchange

A Leibniz triplet has an important property related to least common multiple:

Theorem 5. *In a Leibniz triplet, the vertical pair* $[\beta_{nk};\ \alpha_{nk}]$ *and the horizontal pair* $[\beta_{nk};\ \gamma_{nk}]$ *both share the same least common multiple.*

$\vdash \mathsf{lcm}\ \beta_{nk}\ \alpha_{nk} = \mathsf{lcm}\ \beta_{nk}\ \gamma_{nk}$

Proof. Let $a = \alpha_{nk}$, $b = \beta_{nk}$, $c = \gamma_{nk}$. Recall from Theorem 4 that: $ab = c(b - a)$.

$\mathsf{lcm}\ b\ c$	
$= bc \div \mathsf{gcd}(b, c)$	by definition
$= abc \div (a \times \mathsf{gcd}(b, c))$	introduce factor a above and below division
$= bac \div \mathsf{gcd}(ab, ca)$	by common factor a, commutativity
$= bac \div \mathsf{gcd}(c(b - a), ca)$	by Leibniz triplet property, Theorem 4
$= bac \div (c \times \mathsf{gcd}(b{-}a, a))$	extract common factor c
$= ba \div \mathsf{gcd}(b, a)$	apply GCD subtraction and cancel factor c
$= \mathsf{lcm}\ b\ a$	by definition.

□

row n \ column k	$k = 0,$	$k = 1,$	$k = 2,$	$k = 3,$	$k = 4,$	$k = 5,$	$k = 6,$	$\cdots$
$n = 0$	1							
$n = 1$	2	2						
$n = 2$	3	6	3					
$n = 3$	4	12	12	4				
$n = 4$	5	20	30	20	5			
$n = 5$	6	30	60	60	30	6		
$n = 6$	7	42	105	140	105	42	7	

Table 3. A column and a row intersecting at a left boundary entry of Denominator Triangle

We shall make good use of this LCM invariance through swapping vertical and horizontal pairs in Leibniz triplets to establish an "enlarged" L-shaped LCM invariance involving columns and rows, as shown in Table 3. Theorem 1 will be deduced from this extended LCM invariance.

4 Paths Through Triangles

Our theorem requires us to capture the notion of the least common multiple of a list of elements (a path within the Denominator Triangle). We formalize paths as lists of numbers, without requiring the path to be connected. However, the paths we work with will be connected and include (refer to Table 3):

- $(\mathcal{L}_{\text{down}}\ n)$: the list $[1\ ..\ n + 1]$, which happens to be the first $n + 1$ elements of the leftmost column of the Denominator Triangle, reading down;
- $(\mathcal{L}_{\text{up}}\ n)$: the reverse of $\mathcal{L}_{\text{down}}\ n$, or the leftmost column of the triangle reading up; and
- $(\mathcal{L}_{\text{row}}\ n)$: the n-th row of the Denominator Triangle, reading from the left.

Then, due to the possibility of LCM exchange within a Leibniz triplet (Theorem 5), we can prove the following:

Theorem 6. *In the Denominator Triangle, consider the first element (at left boundary) of the n-th row. Then the least common multiple of the column of elements above it is equal to the least common multiple of elements in its row.*

$\vdash$ list_lcm $(\mathcal{L}_{\text{down}}\ n)$ = list_lcm $(\mathcal{L}_{\text{row}}\ n)$

The proof is done *via* a kind of zig-zag transformation, see Figure 2. In the Denominator Triangle, we represent the entries for LCM consideration as a path of black discs, and indicate the Leibniz triplets by discs marked with small gray dots. Recall that, by Theorem 5, the vertical pair of a Leibniz triplet can be swapped with its horizontal pair without affecting the least common multiple.

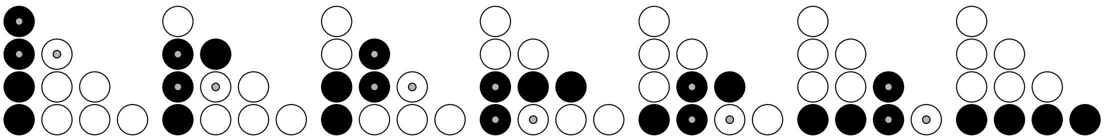

Fig. 2. Transformation of a path from vertical to horizontal in the Denominator Triangle, stepping from left to right. The path is indicated by entries with black discs. The 3 gray-dotted discs in L-shape indicate the Leibniz triplet, which allows LCM exchange. Each step preserves the overall LCM of the path.

It takes a little effort to formalize such a transformation. We use the following approach in HOL4.

4.1 Zig-zag Paths

If a path happens to have a vertical pair, we can match the vertical pair with a Leibniz triplet and swap with its horizontal pair to form another path, its zig-zag equivalent, which keeps the list LCM of the path.

Definition 2. *Zig-zag paths are those transformable by a Leibniz triplet.*

$\vdash p_1 \leadsto p_2 \iff$
$\quad \exists n\ k\ x\ y.\ p_1 = x \frown [\beta_{nk};\ \alpha_{nk}] \frown y \land p_2 = x \frown [\beta_{nk};\ \gamma_{nk}] \frown y$

Basic properties of zig-zag paths are:

Theorem 7. *Zig-zag path properties.*

$\vdash p_1 \leadsto p_2 \Rightarrow \forall x.\ [x] \frown p_1 \leadsto [x] \frown p_2$	zig-zag a congruence wrt (::)
$\vdash p_1 \leadsto p_2 \Rightarrow$ list_lcm p_1 = list_lcm p_2	preserving LCM by exchange *via* triplet

4.2 Wriggle Paths

A path can *wriggle* to another path if there are zig-zag paths in between to facilitate the transformation. Thus, wriggling is the reflexive and transitive closure of zig-zagging, giving the following:

Theorem 8. *Wriggle path properties.*

$\vdash p_1 \leadsto^* p_2 \Rightarrow \forall x.\ [x] \frown p_1 \leadsto^* [x] \frown p_2$	wriggle a congruence wrt (::)
$\vdash p_1 \leadsto^* p_2 \Rightarrow$ list_lcm p_1 = list_lcm p_2	preserves LCM by zig-zags

4.3 Wriggling Inductions

We use wriggle paths to establish a key step[2]:

Theorem 9. *In the Denominator Triangle, a left boundary entry with the entire row above it can wriggle to its own row.*

$\vdash\ [\,\mathcal{L}\ (n + 1)\ 0\,] \frown \mathcal{L}_{\text{row}}\ n \rightsquigarrow^{*} \mathcal{L}_{\text{row}}\ (n + 1)$

Proof. We prove a more general result by induction, with the step case given by the following lemma:

$$\begin{aligned}&\vdash k \le n \Rightarrow\\ &\quad \mathsf{TAKE}\ (k + 1)\ (\mathcal{L}_{\text{row}}\ (n + 1)) \frown \mathsf{DROP}\ k\ (\mathcal{L}_{\text{row}}\ n) \rightsquigarrow\\ &\quad \mathsf{TAKE}\ (k + 2)\ (\mathcal{L}_{\text{row}}\ (n + 1)) \frown \mathsf{DROP}\ (k + 1)\ (\mathcal{L}_{\text{row}}\ n)\end{aligned}$$

where the list operators TAKE and DROP extract, respectively, prefixes and suffixes of our paths.
In other words: in the Denominator Triangle, the two partial rows $\mathsf{TAKE}\ (k + 1)\ (\mathcal{L}_{\text{row}}\ (n + 1))$ and $\mathsf{DROP}\ k\ (\mathcal{L}_{\text{row}}\ n)$ can zig-zag to a longer prefix of the lower row, with the upper row becoming one entry shorter. This is because there is a Leibniz triplet at the zig-zag point (see, for example, Step 5 of Figure 2), making the zig-zag condition possible. The subsequent induction is on the length of the upper partial row. □

With this key step, we can prove the whole transformation illustrated in Figure 2.

Theorem 10. *In the Denominator Triangle, for any left boundary entry: its upward vertical path wriggles to its horizontal path.*

$\vdash\ \mathcal{L}_{\text{up}}\ n \rightsquigarrow^{*} \mathcal{L}_{\text{row}}\ n$

Proof. By induction on the path length n.
For the basis $n = 0$, both $(\mathcal{L}_{\text{up}}\ 0)$ and $(\mathcal{L}_{\text{row}}\ 0)$ are $[1]$, hence they wriggle trivially.
For the induction step, note that the head of $(\mathcal{L}_{\text{up}}\ (n + 1))$ is $(\mathcal{L}\ (n + 1)\ 0)$. Then,

$$\begin{aligned}&\mathcal{L}_{\text{up}}\ (n + 1) &&\\ =\ &[\mathcal{L}\ (n + 1)\ 0] \frown \mathcal{L}_{\text{up}}\ n && \text{by taking apart head and tail}\\ \rightsquigarrow^{*}\ &[\mathcal{L}\ (n + 1)\ 0] \frown \mathcal{L}_{\text{row}}\ n && \text{by induction hypothesis and tail wriggle (Theorem 8)}\\ \rightsquigarrow^{*}\ &\mathcal{L}_{\text{row}}\ (n + 1) && \text{by key step of wriggling (Theorem 9).}\end{aligned}$$

□

Now we can formally prove the LCM transform of Theorem 6.

$\vdash\ \mathsf{list\_lcm}\ (\mathcal{L}_{\text{down}}\ n) = \mathsf{list\_lcm}\ (\mathcal{L}_{\text{row}}\ n)$

Proof. Applying path wriggling of Theorem 10 in the last step,

$$\begin{aligned}&\mathsf{list\_lcm}\ (\mathcal{L}_{\text{down}}\ n) &&\\ =\ &\mathsf{list\_lcm}\ (\mathcal{L}_{\text{up}}\ n) && \text{by reverse paths keeping LCM}\\ =\ &\mathsf{list\_lcm}\ (\mathcal{L}_{\text{row}}\ n) && \text{by wriggle paths keeping LCM (Theorem 8).}\end{aligned}$$

□

[2] This is illustrated in Figure 2 from the middle (step 4) to the last (step 7).

5 LCM Lower Bound

Using the equality of least common multiples just proved for Theorem 6, here is the proof of Theorem 1:

$\vdash\ 2^n \ \le\ \mathsf{list\_lcm}\ [1\ ..\ n\ +\ 1]$

Proof. Recall from Section 3 that the left vertical boundary of Leibniz's Denominator Triangle consists of consecutive numbers, thus $(\mathcal{L}_{\text{down}}\ \ n) = [1\ ..\ n\ +\ 1]$. Also, the horizontal $(\mathcal{L}_{\text{row}}\ \ n)$ is just a multiple of $(\mathcal{P}_{\text{row}}\ \ n)$ by a factor $(n + 1)$. Therefore,

$\ \ \mathsf{list\_lcm}\ [1\ ..\ n\ +\ 1]$	
$= \mathsf{list\_lcm}\ (\mathcal{L}_{\text{down}}\ \ n)$	as asserted
$= \mathsf{list\_lcm}\ (\mathcal{L}_{\text{row}}\ \ n)$	by LCM transform (Theorem 6)
$= (n\ +\ 1)\ \times\ \mathsf{list\_lcm}\ (\mathcal{P}_{\text{row}}\ \ n)$	by LCM common factor
$= \mathsf{LENGTH}\ (\mathcal{P}_{\text{row}}\ \ n)\ \times\ \mathsf{list\_lcm}\ (\mathcal{P}_{\text{row}}\ \ n)$	by length of horizontal row
$\ge \mathsf{SUM}\ (\mathcal{P}_{\text{row}}\ \ n)$	by Theorem 2
$= 2^n$	by binomial sum (Theorem 3).

□

6 Conclusion

We have proved a lower bound for the least common multiple of consecutive numbers, using an interesting application of Leibniz's Triangle in denominator form. By elementary reasoning over natural numbers and lists, we have not just mechanized what we believe to be a cute proof, but now have a result that will be useful in our ongoing work on the mechanization of the AKS algorithm.

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