Bounding LCM with Triangles — Behind the Scenes

How the Proof becomes a Pearl

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Recap

AKS mechanisation

PRIMES is in P

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Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

We will need the following simple fact about the lcm of first m numbers (see, e.g., [Nai82] for a proof).

Lemma 3.1. *Let $LCM(m)$ denote the lcm of first m numbers. For $m \geq 7$:*

$$LCM(m) \geq 2^m.$$

Need to formalize this LCM lemma, but not using Nair's integral-sum.

Math Stack Exchange

Google search leads to Leibniz's Harmonic Triangle.

Is there a direct proof of this lcm identity?

26

7

The identity

$$(n+1)\operatorname{lcm}\left(\binom{n}{0},\binom{n}{1},\dots\binom{n}{n}\right) = \operatorname{lcm}(1,2,\dots n+1)$$

is probably not well-known. The only way I know how to prove it is by using Kummer's theorem that the power of p dividing $\binom{a+b}{a}$ is the number of carries needed to add a and b in base p. Is there a more direct proof, e.g. by showing that each side divides the other?

(number-theory) (binomial-coefficients)

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3 Answers

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19

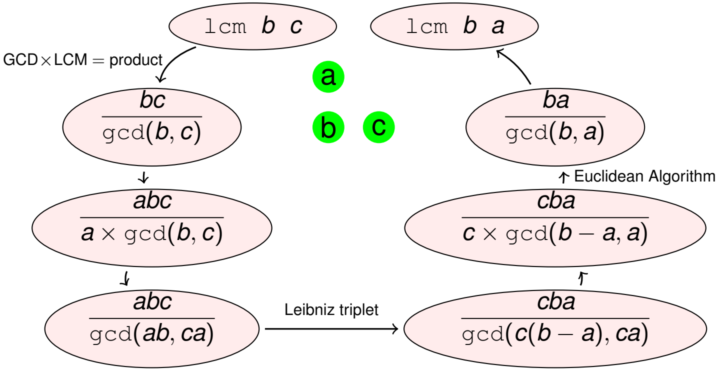
Consider *Leibniz harmonic triangle* — a table that is like «Pascal triangle reversed»: on it's sides lie numbers $\frac{1}{n}$ and each number is the sum of two beneath it (see the picture).

One can easily proove by induction that m-th number in n-th row of Leibniz triangle is $\frac{1}{\dots}$.

Key Property

Theorem (LCM Exchange)

For a Leibniz triplet $\{a, b, c\}$, $\texttt{lcm}\ b\ c = \texttt{lcm}\ b\ a.$



For a Leibniz triplet $\{a, b, c\}$, $\quad ab = c(b-a)$.

Clever Idea

	1					
	2	2				
	3	6	3			
	4	12	12	4		
	5	20	30	20	5	
	6	30	60	60	30	6
$6\times$	1	5	10	10	5	1

Theorem (Lower Bound for the LCM of any list ℓ)

For a list ℓ of positive numbers, $\texttt{SUM}\ \ell \le \texttt{LENGTH}\ \ell \times \texttt{list\_lcm}\ \ell$.

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- Applying theorem to vertical list . . . a disappointing lower bound.
- Applying theorem to horizontal list . . . ??
  This will work because both lists have the same LCM!

The Journey

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  - May not get Q1442: Is there a Direct Proof of this LCM identity?

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  - Leibniz harmonic triangle
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At some point, need to stop surfing and DIY.

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Consider *Leibniz harmonic triangle* — a table that is like «Pascal triangle reversed»: on it's sides lie numbers $\frac{1}{n}$ and each number is the sum of two beneath it (see the picture).

One can easily proove by induction that m-th number in n-th row of Leibniz triangle is $\frac{1}{(n+1)\binom{n}{m}}$.

So LHS of our identity is just lcd of fractions in n-th row of the triangle.

But it's not hard to see that any such number is an integer linear combination of fractions on triangle's sides (i.e. $1/1, 1/2, \dots, 1/n$) — and vice versa. So LHS is equal to $lcd(1/1, \dots, 1/n)$ — and that is exactly RHS.

Induction Pattern

```
(* LCM Lemma

(n+1) lcm (C(n,0) to C(n,n)) = lcm (1 to (n+1))

m-th number in the n-th row of Leibniz triangle is:  1/ (n+1)C(n,m)

So LHS = lcd (1/1, 1/2, 1/3, ..., 1/n) = RHS = lcm (1,2,3, ..., (n+1)).

0-th row:                   1
1-st row:               1/2   1/2
2-nd row:            1/3   1/6   1/3
3-rd row:        1/4   1/12   1/12   1/4
4-th row: 1/5  1/20  1/30  1/20  1/5

4-th row: 1/5 C(4,m), C(4,m) = 1 4 6 4 1, hence 1/5 1/20 1/30 1/20 1/5
  lcd (1/5 1/20 1/30 1/20 1/5)
= lcm (5, 20, 30, 20, 5)
= lcm (5 C(4,0), 5 C(4,1), 5 C(4,2), 5 C(4,3), 5 C(4,4))
= 5 lcm (C(4,0), C(4,1), C(4,2), C(4,3), C(4,4))
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```

- How to prove the identity by induction? Need a pattern.
- Assuming the identity, does it lead to the lower bound?

Finding Pattern

```
Theorem: In the Multiples Triangle, the vertical-lcm = horizontal-lcm.
i.e.     lcm (1, 2, 3) = lcm (3, 6, 3) = 6
         lcm (1, 2, 3, 4) = lcm (4, 12, 12, 4) = 12
         lcm (1, 2, 3, 4, 5) = lcm (5, 20, 30, 20, 5) = 60
         lcm (1, 2, 3, 4, 5, 6) = lcm (6, 30, 60, 60, 30, 6) = 60
Proof: With reference to Leibniz's Triangle, note: term = left-up - left
  lcm (5, 20, 30, 20, 5)
= lcm (5, 20, 30)                    by reduce repetition
= lcm (5, d(1/20), d(1/30))          by denominator of fraction
= lcm (5, d(1/4 - 1/5), d(1/30))     by term = left-up - left
= lcm (5, lcm(4, 5), d(1/12 - 1/20))     by denominator of fraction subtraction
= lcm (5, 4, lcm(12, 20))                by lcm (a, lcm (a, b)) = lcm (a, b)
= lcm (5, 4, lcm(d(1/12), d(1/20)))      to fraction again
= lcm (5, 4, lcm(d(1/3 - 1/4), d(1/4 - 1/5)))   by Leibniz's Triangle
= lcm (5, 4, lcm(lcm(3,4),     lcm(4,5)))     by fraction subtraction denominator
= lcm (5, 4, lcm(3, 4, 5))                    by lcm merge
= lcm (5, 4, 3)                               merge again
= lcm (5, 4, 3, 2)                            by lcm include factor (!!!)
= lcm (5, 4, 3, 2, 1)                         by lcm include 1
```

A sample of my investigation, by examples.

Promising Result

```

  lcm (1 to 5)                = 1x2x3x4x5/2 = 60
= 5 lcm (1 4 6 4 1)           = 5 x 12
=  lcm (1 4 6 4 1)            --> unfold 5x to add 5 times
 + lcm (1 4 6 4 1)
 + lcm (1 4 6 4 1)
 + lcm (1 4 6 4 1)
 + lcm (1 4 6 4 1)
>= 1 + 4 + 6 + 4 + 1          --> pick one of each 5 C(n,m), i.e. diagonal
= (1 + 1)^4                   --> fold back binomial
= 2^4                         = 16

Actually, can take 5 lcm (1 4 6 4 1) >= 5 x 6 = 30,
but this will need estimation of C(n, n/2), or C(2n,n), i.e. Stirling's formula.

Theorem: lcm (x y z) >= x  or lcm (x y z) >= y  or lcm (x y z) >= z

```

Figure out that the LCM identity leads to the desired lower bound.

Hit an Idea

```
(* The Idea:
                                                               b
Actually, lcm a b = lcm b c = lcm c a      for    a c  in Leibniz Triangle.
The only relationship is: c = ab/(a - b), or ab = c(a - b).

Is this a theorem:  ab = c(a - b)  ==> lcm a b = lcm b c = lcm c a
Or in fractions,   1/c = 1/b - 1/a ==> lcm a b = lcm b c = lcm c a ?

  lcm a b
= a b / (gcd a b)
= c(a - b) / (gcd a (a - b))
= ac(a - b) / gcd a (a-b) / a
= lcm (a (a-b)) c / a
= lcm (ca c(a-b)) / a
= lcm (ca ab) / a
= lcm b c

  lcm b c
= b c / gcd b c
= a b c / gcd a*b a*c
= a b c / gcd c*(a-b) c*a
= a b / gcd (a-b) a
= a b / gcd b a
= lcm (a b)
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```

Focus on a triplet . . .

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= lcm (a (a-b)) c / a
= lcm (ca c(a-b)) / a
= lcm (ca ab) / a
= lcm b c

  lcm b c
= b c / gcd b c
= a b c / gcd a*b a*c
= a b c / gcd c*(a-b) c*a
= a b / gcd (a-b) a
= a b / gcd b a
= lcm (a b)
= lcm a b
```

Focus on a triplet … hope: $\texttt{lcm}\ a\ b = \texttt{lcm}\ b\ c = \texttt{lcm}\ c\ a$.

Voliá

```
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                                                    b
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  lcm a c
= a c / gcd a c
= a b c / gcd b*a b*c
= a b c / gcd c*(a-b) b*c
= a b / gcd (a-b) b
= a b / gcd a b
= lcm a b

Yes!

This is now in LCM_EXCHANGE:
val it = |- !a b c. (a * b = c * (a - b)) ==> (lcm a b = lcm a c): thm
*)
```

Success!

Polishing

Done and Dusted

- Once the key is proved (SourceTree #1200), goal is within reach.
- Had the picture of path transform, zig-zag and wriggle, for induction.
- Just establish the LCM lower bound by brute-force induction (#1211).

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```
   Transform from Vertical LCM to Horizontal LCM:
   leibniz_lcm_shift_one      |- !n k. k <= n ==>
                                 (lcm (list_lcm (TAKE (SUC k) (leibniz_horizontal (SUC n))))
                                      (list_lcm (DROP k (leibniz_horizontal n))) =
                                  lcm (list_lcm (TAKE (SUC (SUC k)) (leibniz_horizontal (SUC n))))
                                      (list_lcm (DROP (SUC k) (leibniz_horizontal n))))
   leibniz_lcm_shift          |- !n k. k <= SUC n ==>
                                 (lcm (list_lcm (TAKE (SUC k) (leibniz_horizontal (SUC n))))
                                      (list_lcm (DROP k (leibniz_horizontal n))) =
                                  lcm (SUC (SUC n)) (list_lcm (leibniz_horizontal n)))
   leibniz_horizontal_lcm     |- !n. list_lcm (leibniz_horizontal (SUC n)) =
                                     lcm (SUC (SUC n)) (list_lcm (leibniz_horizontal n))
   leibniz_lcm_property       |- !n. list_lcm (leibniz_vertical n) = list_lcm (leibniz_horizontal n)

   Binomial Horizontal List:
   binomial_horizontal_def    |- !n. binomial_horizontal n = GENLIST (binomial n) (SUC n)
   binomial_horizontal_0      |- binomial_horizontal 0 = [1]
   binomial_horizontal_len    |- !n. LENGTH (binomial_horizontal n) = n + 1
   binomial_horizontal_pos    |- !n. EVERY (\x. 0 < x) (binomial_horizontal n)
   binomial_horizontal_sum    |- !n. SUM (binomial_horizontal n) = 2 ** n

   Lower Bound of Leibniz LCM:
   leibniz_alt                |- !n. leibniz n = (\k. (n + 1) * k) o binomial n
   leibniz_horizontal_alt     |- !n. leibniz_horizontal n = MAP (\k. (n + 1) * k) (binomial_horizontal n)
   leibniz_horizontal_lcm_alt          |- !n. list_lcm (leibniz_horizontal n) =
                                              (n + 1) * list_lcm (binomial_horizontal n)
   leibniz_horizontal_lcm_lower_bound  |- !n. 2 ** n <= list_lcm (leibniz_horizontal n)
   leibniz_vertical_lcm_lower_bound    |- !n. 2 ** n <= list_lcm (leibniz_vertical n)
*)
```

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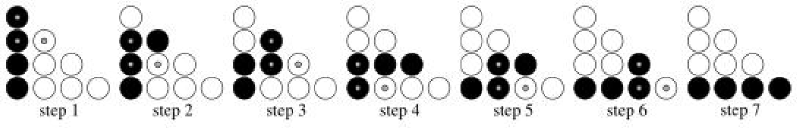

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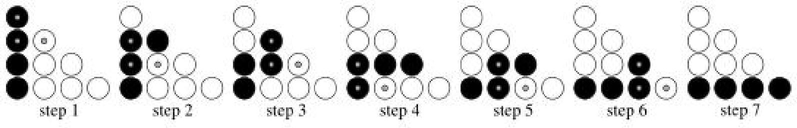

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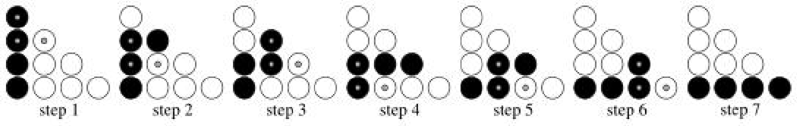

- A good picture, but the proof script is bad — heaps of induction.
- Realize that zig-zags and wriggles are implicit in current proofs.
- Replace brute-force induction bt explicit zig-zag and wriggle paths.

Major Changes

- Formalize in HOL4: path transform, zig-zag and wriggle.
- Reformulate the proofs based on such concepts (#1531).

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   Using Triplet and Paths:
   leibniz_zigzag_def     |- !path1 path2. path1 zigzag path2 <=>
                             ?n k part1 part2. (path1 = part1 ++ [tri_b] ++ [tri_a] ++ part2) /\
                                               (path2 = part1 ++ [tri_b] ++ [tri_c] ++ part2)
   leibniz_wriggle_def    |- !path1 path2. path1 wriggle path2 <=>
                             ?m f. (path1 = f 0) /\ (path2 = f m) /\ !k. k < m ==> f k zigzag f (SUC k)
   leibniz_lcm_triple     |- !n k. lcm tri_b tri_a = lcm tri_b tri_c
   list_lcm_zigzag        |- !path1 path2. path1 zigzag path2 ==> (list_lcm path1 = list_lcm path2)
   list_lcm_wriggle       |- !path1 path2. path1 wriggle path2 ==> (list_lcm path1 = list_lcm path2)
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   leibniz_zigzag_tail       |- !path1 path2. path1 zigzag path2 ==> !x. [x] ++ path1 zigzag [x] ++ path2
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   leibniz_horizontal_wriggle
                          |- !n. [SUC (SUC n)] ++ leibniz_horizontal n wriggle leibniz_horizontal (SUC n)

   leibniz_up_0           |- leibniz_up 0 = [1]
   leibniz_up_len         |- !n. LENGTH (leibniz_up n) = SUC n
   leibniz_up_cons        |- !n. leibniz_up (SUC n) = SUC (SUC n)::leibniz_up n
   leibniz_triplet_0      |- leibniz_up 1 zigzag leibniz_horizontal 1
   leibniz_up_wriggle_horizontal  |- !n. 0 < n ==> leibniz_up n wriggle leibniz_horizontal n
   leibniz_lcm_property   |- !n. list_lcm (leibniz_vertical n) = list_lcm (leibniz_horizontal n)
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- Formalize in HOL4: path transform, zig-zag and wriggle.
- Reformulate the proofs based on such concepts (#1531).
- A 12-page draft, with wonderful diagrams, tables and proofs.

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- Re-package diagrams and tables side-by-side, use explicit triplet.

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- Re-package diagrams and tables side-by-side, use explicit triplet.
- Wriggle is the reflexive transitive closure (RTC) of zig-zag (#1567).

Final Touch

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```

   Wriggle Paths in Leibniz Triangle (old):
   leibniz_old_wriggle_def      |- !p1 p2. p1 old_wriggle p2 <=>
                                   ?m f. (p1 = f 0) /\ (p2 = f m) /\ !k. k < m ==> f k zigzag f (SUC k)
   list_lcm_old_wriggle         |- !p1 p2. p1 old_wriggle p2 ==> (list_lcm p1 = list_lcm p2)
   leibniz_zigzag_old_wriggle   |- !p1 p2. p1 zigzag p2 ==> p1 old_wriggle p2
   leibniz_old_wriggle_tail     |- !p1 p2. p1 old_wriggle p2 ==> !x. [x] ++ p1 old_wriggle [x] ++ p2
   leibniz_old_wriggle_trans    |- !p1 p2 p3. p1 old_wriggle p2 /\ p2 old_wriggle p3 ==> p1 old_wriggle p3
   leibniz_horizontal_old_wriggle    |- !n. [leibniz (n + 1) 0] ++ leibniz_horizontal n old_wriggle
                                             leibniz_horizontal (n + 1)

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   leibniz_wriggle_trans     |- !p1 p2 p3. p1 wriggle p2 /\ p2 wriggle p3 ==> p1 wriggle p3

   Back to Milestone Theorem:
   leibniz_triplet_0         |- leibniz_up 1 zigzag leibniz_horizontal 1
   leibniz_up_old_wriggle_horizontal   |- !n. 0 < n ==> leibniz_up n old_wriggle leibniz_horizontal n
   leibniz_lcm_property      |- !n. list_lcm (leibniz_vertical n) = list_lcm (leibniz_horizontal n)

*)
```

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- Last day: can't complete RTC induction. Help!

```

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Reviews

Review #1, Expertise: high

This paper presents a "proof pearl", a short and clever proof that $2^n \leq lcm(1, ..., n+1)$. This is not a trivial result: Nair's proof of this fact was published in 1982, and Google search reveals some recent strengthenings and generalizations, but it seems that there is no published elementary proof of this fact.

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The wording of Theorem 5 *is confusing. [...] How about saying this: [...]*

The reference to the "unrolling" in Section 5 *makes it mysterious, and the proof is needlessly baroque. The argument is simply this: [...]*

Review #2, Expertise: medium

The authors describe a (mechanised) proof of a number-theoretic fact: $2^n \leq lcm(1, ..., n+1)$*. The proof is not new, but the paper is advertised as a pearl.*

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In the past I have reviewed several papers that were advertised as pearls, but that in my opinion were not pearls. That is not the case with this paper. I found the text engaging, and easy to follow. The proof is non-trivial, but the authors made it easy to understand for me, and I thought that the mechanisation was presented at a suitable level of detail.

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I strongly recommend the paper for publication.

Review #3, Expertise: medium

This proof pearl shows a lower bound for the least common multiple of the first n integers [...]

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Although the inequality is quite specific, this paper demonstrates that it is worth to search for elegant proofs rather than to apply the golden hammer of a complicated theory. Indeed, the formalised proof is very elementary compared to the published proofs I know of. The authors have done a good job of bringing together the proof ingredients (which have been known) and explaining the proof idea.

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In summary, I think that this paper makes a nice proof pearl, and I therefore recommend acceptance.

Epilog

Conclusion

This talk is dedicated to

Michael Norrish,

my supervisor.

- Scripts
  `https://bitbucket.org/jhlchan/hol/src/`
  subfolder: `algebra/lib`.
- Paper
  `https://bitbucket.org/jhlchan/hol/downloads`