Least Common Multiples and Triangles

based on ITP 2016: Proof Pearl

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How to play with numbers

Given the numbers: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.
What can you do with them?

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- multiply them: $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 =$ $\texttt{Multiply}[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] = 3628800$

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- combine by GCD: $\texttt{GCD}[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] = 1$

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- combine by GCD: $\texttt{GCD}[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] = 1$
- combine by LCM: $\texttt{LCM}[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] = 2520$

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Theorem

$\vdash \texttt{GCD}[1, 2, 3, \ldots, n] = 1$

True, but not interesting. ☹

Math Formula

Math is about patterns.

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- $\texttt{Add}[1, 2, 3, \ldots, n] = n(n+1)/2$, or
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- $\sqrt{2\pi}\,\sqrt{n}\,(n/e)^n \;\le\; \texttt{Multiply}[1, 2, 3, \ldots, n] \;\le\; e\,\sqrt{n}\,(n/e)^n$
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  useful for my PhD work.

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Theorem

$\vdash 2^n \leq \texttt{LCM}[1, 2, 3, \ldots, (n + 1)]$

True, and interesting! ☺

LCM of a List

`LCM` [1, 2]
`LCM` [1, 2, 3]
`LCM` [1, 2, 3, 4]
`LCM` [1, 2, 3, 4, 5]
`LCM` [1, 2, 3, 4, 5, 6]

LCM of a List

$\texttt{LCM}\,[1, 2] = 2$
$\texttt{LCM}\,[1, 2, 3] = 6$
$\texttt{LCM}\,[1, 2, 3, 4] = 12$
$\texttt{LCM}\,[1, 2, 3, 4, 5] = 60$
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Note that the LCM is a multiple of each element,
or each element is less than the overall LCM.

LCM of a List

$$
\begin{aligned}
&\texttt{LCM}\,[1, 2] = 2 && \geq 2^1 = 2 \\
&\texttt{LCM}\,[1, 2, 3] = 6 && \geq 2^2 = 2 \times 2 = 4 \\
&\texttt{LCM}\,[1, 2, 3, 4] = 12 && \geq 2^3 = 2 \times 2 \times 2 = 8 \\
&\texttt{LCM}\,[1, 2, 3, 4, 5] = 60 && \geq 2^4 = 2 \times 2 \times 2 \times 2 = 16 \\
&\texttt{LCM}\,[1, 2, 3, 4, 5, 6] = 60 && \geq 2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32
\end{aligned}
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$\vdash \texttt{LCM}\ [1\ ..\ n + 1] \geq 2^n$

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How to prove this interesting result?

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Note that the LCM is a multiple of each element,
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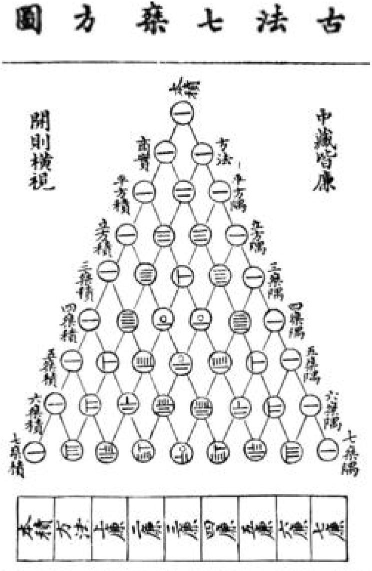
Theorem

$\vdash$ LCM [1 .. n + 1] $\geq 2^n$

How to prove this interesting result? ... use Triangles! ⌑

Yang Hui's Triangle



Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

- Each boundary entry: always 1.

Pascal's Triangle

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1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
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- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

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1 2 1
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Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

- Each boundary entry: always 1.
- Each inside entry: sum of two immediate parents.

Sum of the *n*-th row:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Pascal's Triangle – Row Sum

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Meaning of:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Pascal's Triangle – Row Sum

						1							Row sum here.
					1		1						
				1		2		1					
			1		3		3		1				
		1		4		6		4		1			
	1		5		10		10		5		1		
1		6		15		20		15		6		1	

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Pascal's Triangle – Row Sum

						1						
					1	+	1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Row sum here.

$= 2 = 2^1$

Meaning of:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Pascal's Triangle – Row Sum

							1							Row sum here.
						1	+	1						$= 2 = 2^1$
					1	+	2	+	1					$= 4 = 2^2$
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	

Meaning of:

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Pascal's Triangle – Row Sum

1	Row sum here.
$1 + 1$	$= 2 = 2^1$
$1 + 2 + 1$	$= 4 = 2^2$
$1 + 3 + 3 + 1$	$= 8 = 2^3$
1 4 6 4 1	
1 5 10 10 5 1	
1 6 15 20 15 6 1	

Meaning of:

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$1 + 4 + 6 + 4 + 1$	$= 16 = 2^4$
1 5 10 10 5 1	
1 6 15 20 15 6 1	

Meaning of:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Pascal's Triangle – Row Sum

$$
\begin{array}{ccccccccccccccl}
 & & & & & & 1 & & & & & & & & \text{Row sum here.} \\
 & & & & & 1 & + & 1 & & & & & & & = 2 = 2^1 \\
 & & & & 1 & + & 2 & + & 1 & & & & & & = 4 = 2^2 \\
 & & & 1 & + & 3 & + & 3 & + & 1 & & & & & = 8 = 2^3 \\
 & & 1 & + & 4 & + & 6 & + & 4 & + & 1 & & & & = 16 = 2^4 \\
 & 1 & + & 5 & + & 10 & + & 10 & + & 5 & + & 1 & & & = 32 = 2^5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & &
\end{array}
$$

Meaning of:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Pascal's Triangle – Row Sum

$$
\begin{array}{c}
1 \\
1 + 1 \\
1 + 2 + 1 \\
1 + 3 + 3 + 1 \\
1 + 4 + 6 + 4 + 1 \\
1 + 5 + 10 + 10 + 5 + 1 \\
1 + 6 + 15 + 20 + 15 + 6 + 1
\end{array}
\quad
\begin{array}{c}
\text{Row sum here.} \\
= 2 = 2^1 \\
= 4 = 2^2 \\
= 8 = 2^3 \\
= 16 = 2^4 \\
= 32 = 2^5 \\
= 64 = 2^6
\end{array}
$$

Meaning of:

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

Triangle Pattern

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}
$$

From symmetrical form to vertical-horizontal form.

Leibniz's Denominator Triangle

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Leibniz's Denominator Triangle

1

$\Leftarrow \times 1$

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Leibniz's Denominator Triangle

1		$\Leftarrow \times 1$	1								
2	2	$\Leftarrow \times 2$	1	1							
			1	2	1						
			1	3	3	1					
			1	4	6	4	1				
			1	5	10	10	5	1			
			1	6	15	20	15	6	1		

Leibniz's Denominator Triangle

1			$\Leftarrow \times 1$	1							
2	2		$\Leftarrow \times 2$	1	1						
3	6	3	$\Leftarrow \times 3$	1	2	1					
				1	3	3	1				
				1	4	6	4	1			
				1	5	10	10	5	1		
				1	6	15	20	15	6	1	

Leibniz's Denominator Triangle

1			
2	2		
3	6	3	
4	12	12	4

$\Leftarrow \times 1$	1						
$\Leftarrow \times 2$	1	1					
$\Leftarrow \times 3$	1	2	1				
$\Leftarrow \times 4$	1	3	3	1			
	1	4	6	4	1		
	1	5	10	10	5	1	
	1	6	15	20	15	6	1

Leibniz's Denominator Triangle

1					$\Leftarrow \times 1$	1						
2	2				$\Leftarrow \times 2$	1	1					
3	6	3			$\Leftarrow \times 3$	1	2	1				
4	12	12	4		$\Leftarrow \times 4$	1	3	3	1			
5	20	30	20	5	$\Leftarrow \times 5$	1	4	6	4	1		
						1	5	10	10	5	1	
						1	6	15	20	15	6	1

Leibniz's Denominator Triangle

1							$\Leftarrow \times 1$	1						
2	2						$\Leftarrow \times 2$	1	1					
3	6	3					$\Leftarrow \times 3$	1	2	1				
4	12	12	4				$\Leftarrow \times 4$	1	3	3	1			
5	20	30	20	5			$\Leftarrow \times 5$	1	4	6	4	1		
6	30	60	60	30	6		$\Leftarrow \times 6$	1	5	10	10	5	1	
								1	6	15	20	15	6	1

Leibniz's Denominator Triangle

1							$\Leftarrow \times 1$	1								
2	2						$\Leftarrow \times 2$	1	1							
3	6	3					$\Leftarrow \times 3$	1	2	1						
4	12	12	4				$\Leftarrow \times 4$	1	3	3	1					
5	20	30	20	5			$\Leftarrow \times 5$	1	4	6	4	1				
6	30	60	60	30	6		$\Leftarrow \times 6$	1	5	10	10	5	1			
7	42	105	140	105	42	7	$\Leftarrow \times 7$	1	6	15	20	15	6	1		

Leibniz's Denominator Triangle

1							$\Leftarrow \times 1$	1						
2	2						$\Leftarrow \times 2$	1	1					
3	6	3					$\Leftarrow \times 3$	1	2	1				
4	12	12	4				$\Leftarrow \times 4$	1	3	3	1			
5	20	30	20	5			$\Leftarrow \times 5$	1	4	6	4	1		
6	30	60	60	30	6		$\Leftarrow \times 6$	1	5	10	10	5	1	
7	42	105	140	105	42	7	$\Leftarrow \times 7$	1	6	15	20	15	6	1

Multiplying each row by the number of elements of that row, the result is Leibniz's Triangle, in denominator form.

Consecutive LCM

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

How does this triangle relate to our goal?

Consecutive LCM

1					
2	2				
3	6	3			
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5	20	30	20	5	
6	30	60	60	30	6

How does this triangle relate to our goal?
We shall show this: $\mathtt{LCM}$ (vertical column) $=$ $\mathtt{LCM}$ (horizontal row).

Consecutive LCM

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

How does this triangle relate to our goal?
We shall show this: $\texttt{LCM}$ (vertical column) $=$ $\texttt{LCM}$ (horizontal row).

This is LCM exchange for big-L,
and $\texttt{LCM}$ (horizontal row) is easier to estimate.

Triplets in Triangle

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

Building blocks of Leibniz's triangle: the small-Ls, called triplets.

Triplets in Triangle

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

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2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

Building blocks of Leibniz's triangle: the small-Ls, called triplets.

- Each triplet has a vertical pair and a horizontal pair.

Triplets in Triangle

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	20	5	
6	30	60	60	30	6

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- Each triplet has a vertical pair and a horizontal pair.
- We will show: $\texttt{LCM}$ (vertical pair) $=$ $\texttt{LCM}$ (horizontal pair)

Triplets in Triangle

1					
2	2				
3	6	3			
4	12	12	4		
5	20	30	30	5	
6	30	60	60	60	6

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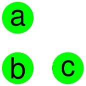
This is LCM exchange for small-L.

Leibniz Triplet Property

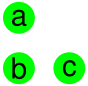


For a Leibniz triplet $\{a, b, c\}$, $\quad ab = c(b - a)$.

Leibniz Triplet Property

Theorem (LCM Exchange)

For a Leibniz triplet $\{a, b, c\}$, `lcm` b c = `lcm` b a.

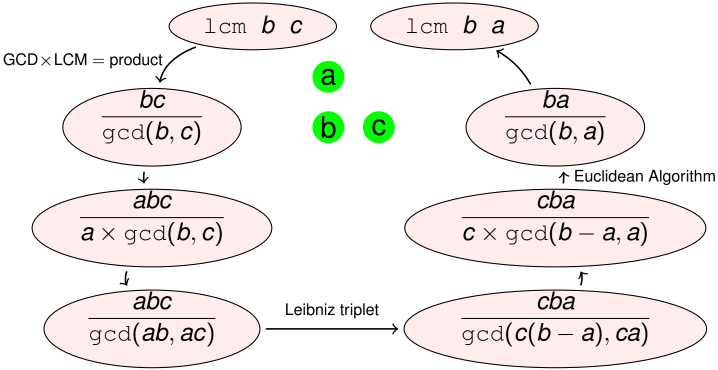


For a Leibniz triplet $\{a, b, c\}$, $ab = c(b - a)$.

Leibniz Triplet Property

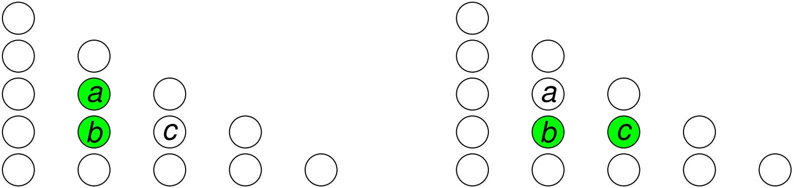
Theorem (LCM Exchange)

For a Leibniz triplet $\{a, b, c\}$, $\texttt{lcm}\ b\ c = \texttt{lcm}\ b\ a$.



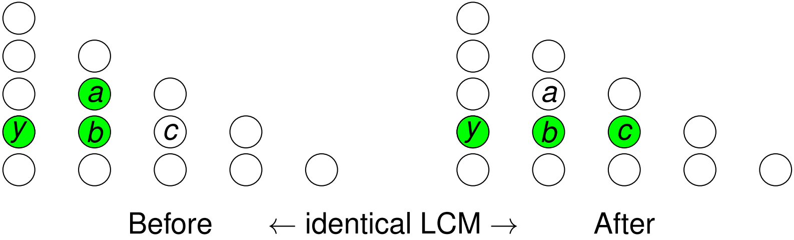
For a Leibniz triplet $\{a, b, c\}$, $ab = c(b - a)$.

Zig-zag Paths



Before ← identical LCM → After

Zig-zag Paths



- Arms of a Leibniz triplet extend to paths, keeping overall LCM.

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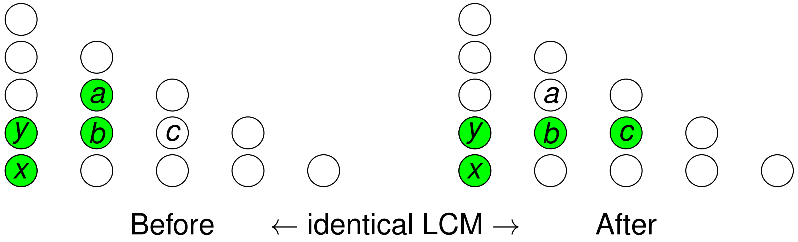

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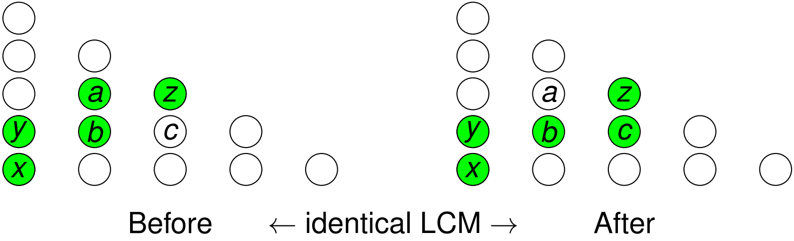

- Arms of a Leibniz triplet extend to paths, keeping overall LCM.
- A path can zig-zag to another by a suitable Leibniz triplet.

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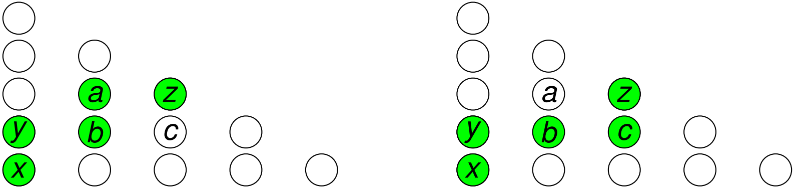

Before $\leftarrow$ identical LCM $\rightarrow$ After

- Arms of a Leibniz triplet extend to paths, keeping overall LCM.
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By Leibniz triplet property,

Theorem

$\vdash p_1 \rightsquigarrow p_2 \Rightarrow \mathit{LCM}\ p_1 = \mathit{LCM}\ p_2$

Wriggle Paths

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

Wriggle Paths

1				
2	2			
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- Transform a path by successive zig-zags keeps overall LCM.

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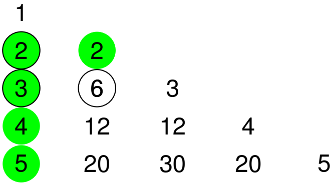

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Wriggle Paths

1				
2	**2**			
3	**6**	3		
4	12	12	4	
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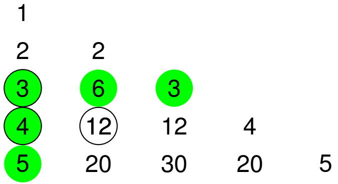

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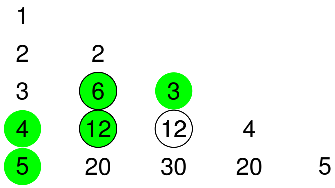

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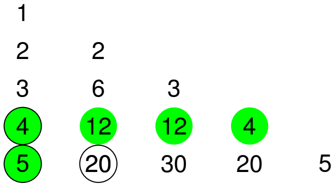

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By Leibniz triplet property,

Theorem

$\vdash p_1 \rightsquigarrow^* p_2 \Rightarrow \mathit{LCM}\ p_1 = \mathit{LCM}\ p_2$

Proof Idea for $2^n \leq \texttt{LCM}[1, \ldots, (n+1)]$

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

`LCM [1, 2, 3, 4, 5]`

Proof Idea for $2^n \leq \texttt{LCM}[1, \ldots, (n+1)]$

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

$$\begin{aligned} & \texttt{LCM [1, 2, 3, 4, 5]} \\ = \ & \texttt{LCM [5, 20, 30, 20, 5]} \end{aligned}$$

by wriggling path transform

Proof Idea for $2^n \leq \texttt{LCM}[1, \ldots, (n+1)]$

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

$$
\begin{aligned}
& \texttt{LCM}\ [1,\ 2,\ 3,\ 4,\ 5] & \\
= \ & \texttt{LCM}\ [5,\ 20,\ 30,\ 20,\ 5] & \text{by wriggling path transform} \\
= \ & 5 \times \texttt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] & \text{taking out common factor}
\end{aligned}
$$

Proof Idea for $2^n \leq \mathtt{LCM}[1, \ldots, (n+1)]$

1				
2	2			
3	6	3		
4	12	12	4	
5	20	30	20	5

$\quad$ LCM [1, 2, 3, 4, 5]

= LCM [5, 20, 30, 20, 5] $\quad$ by wriggling path transform

= 5 × LCM [1, 4, 6, 4, 1] $\quad$ taking out common factor

= LCM [1, 4, 6, 4, 1] $\quad$ multiply = repeated add

\+ LCM [1, 4, 6, 4, 1]

\+ LCM [1, 4, 6, 4, 1]

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\+ LCM [1, 4, 6, 4, 1] $\quad$ LCM = least common multiple

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= \ & \quad \texttt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] && \text{multiply = repeated add} \\
& + \texttt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] \\
& + \texttt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] \\
& + \texttt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] \\
& + \texttt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] && \text{LCM = least common multiple} \\
\geq \ & 1 + 4 + 6 + 4 + 1 && \text{picking diagonal elements}
\end{aligned}$$

Proof Idea for $2^n \leq \mathtt{LCM}[1, \ldots, (n+1)]$

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2	2			
3	6	3		
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$$
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= \ & 5 \times \mathtt{LCM}\ [1,\ 4,\ 6,\ 4,\ 1] & \text{taking out common factor} \\
= \ & \quad \mathtt{LCM}\ [\,\boxed{1},\ 4,\ 6,\ 4,\ 1\,] & \text{multiply = repeated add} \\
& + \mathtt{LCM}\ [\,1,\ \boxed{4},\ 6,\ 4,\ 1\,] & \\
& + \mathtt{LCM}\ [\,1,\ 4,\ \boxed{6},\ 4,\ 1\,] & \\
& + \mathtt{LCM}\ [\,1,\ 4,\ 6,\ \boxed{4},\ 1\,] & \\
& + \mathtt{LCM}\ [\,1,\ 4,\ 6,\ 4,\ \boxed{1}\,] & \text{LCM = least common multiple} \\
\geq \ & 1 + 4 + 6 + 4 + 1 & \text{picking diagonal elements} \\
= \ & (1 + 1)^4 = 2^4 & \text{row sum of Pascal's triangle.}
\end{aligned}
$$

Reference

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