Mechanisation of AKS Algorithm

Part 1 — the Main Theorem

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Basic Primality Test

Theorem (Basic Primality Test.)

$$\vdash 1 < n \wedge \gcd(c, n) = 1 \Rightarrow (\text{prime } n \iff (X + \boldsymbol{c})^n \equiv (X^n + \boldsymbol{c}) \pmod{n}))$$

Proof.

($\Rightarrow$) Apply Freshman's Theorem:

$\text{prime } n \wedge \text{poly p} \wedge \text{poly q} \Rightarrow (\text{p} + \text{q})^n \equiv \text{p}^n + \text{q}^n \pmod{n}$

by identifying p as $\boldsymbol{X}$ and q as $\boldsymbol{c}$,

then apply Fermat's Little Theorem: $\text{prime } n \Rightarrow \boldsymbol{c}^n \equiv \boldsymbol{c} \pmod{n}$

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($\Leftarrow$) Can be shown by properties of binomial coefficients. □

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Given a number $n > 1$, is n a prime?

- Since $\gcd(1, n) = 1$, pick $c = 1$, then this theorem applies.
- Perform one Freshman-Fermat identity check in $\mathbb{Z}_n$, *i.e.*, prime $n \iff (\boldsymbol{X} + \boldsymbol{1})^n \equiv \boldsymbol{X}^n + \boldsymbol{1} \pmod{n}$.

This is a deterministic primality test, however:

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Enter Manindra Agrawal, Neeraj Kayal, and Nitin Saxena (AKS).

AKS twists

The AKS team modifies the Freshman-Fermat identity check:

- Perform the polynomial identity checks in $\pmod{n,\ \boldsymbol{X}^k - \mathbf{1}}$ for some suitably chosen k.
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The AKS result:

- Modifications $\Rightarrow n = p^e$ where prime $p \mid n$ for some exponent e.
- Include a power check: if n is power free, then n must be prime.

The AKS Main Theorem

The AKS primality test is based on the following theorem:

Theorem (The AKS Main Theorem.)

$$
\begin{array}{l}
\vdash \text{prime } n \iff \\
\quad 1 < n \land \text{power\_free } n \land \\
\quad \exists k. \\
\qquad \text{prime } k \land (2(\log n + 1))^2 \le \text{order}_k(n) \land \\
\qquad (\forall j.\ 0 < j \land j \le k \land j < n \Rightarrow \gcd(n,j) = 1) \land \\
\qquad (k < n \Rightarrow \\
\qquad\quad \forall c. \\
\qquad\qquad 0 < c \land c \le 2\sqrt{k}(\log n + 1) \Rightarrow \\
\qquad\qquad\quad (\boldsymbol{X} + \boldsymbol{c})^n \equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n,\ \boldsymbol{X}^k - 1})
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    prime k $\land$ $(2(\log n + 1))^2 \le \mathrm{order}_k(n)$ $\land$
    $(\forall j.\ 0 < j \land j \le k \land j < n \Rightarrow \gcd(n,j) = 1)$ $\land$
    $(k < n \Rightarrow$
      $\forall c.$
        $0 < c \land c \le 2\sqrt{k}(\log n + 1) \Rightarrow$
          $(\boldsymbol{X} + \boldsymbol{c})^n \equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n,\ \boldsymbol{X}^k - 1})$

Note the checks: power free, GCD tests and Polynomial modulo tests.

The AKS Algorithm

The AKS Main theorem corresponds to the following algorithm:

Input: integer $n > 1$.

1. If ($n = b^m$ for some base b with $m > 1$), return `COMPOSITE`.
2. Search for a prime k satisfying $\text{order}_k(n) \geq (2(\log n + 1))^2$.
3. For each ($j = 1$ to k) if ($j = n$) break, else if ($\gcd(j, n) \neq 1$), return `COMPOSITE`.
4. If ($k \geq n$), return `PRIME`.
5. For each ($c = 1$ to ℓ) where $\ell = 2\sqrt{k}(\log n + 1)$, if $(\boldsymbol{X} + \boldsymbol{c})^n \not\equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n, \boldsymbol{X}^k - 1}$, return `COMPOSITE`.
6. return `PRIME`.

Mechanisation of AKS Algorithm

AKS Algorithm

- Found in August 2002 by the team: Agrawal, Kayal and Saxena.
- The first deterministic polynomial-time primality-testing algorithm.
- Correctness of algorithm relies on a new theorem in finite fields.

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Its Mechanisation

- Show the correctness of AKS Main Theorem in a theorem-prover.
- Establish upper bounds on the parameters of AKS algorithm.
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- Show the correctness of AKS Main Theorem in a theorem-prover.
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These correspond to Part 1, Part 2, and Part 3 of our work.

Easy and Hard

The if-part ($\Rightarrow$) is easy:

- prime $n \Rightarrow \gcd(c, n) = 1$ for $1 \leq c \leq \ell$, when $\ell < n$.
- Thus the if-part of Basic Primality Test applies, *i.e.*,
  $(\boldsymbol{X} + \boldsymbol{c})^n \equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n}$ for $1 \leq c \leq \ell$.
- or, $(\boldsymbol{X} + \boldsymbol{c})^n \equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n,\ \boldsymbol{X}^k - 1}$ for $1 \leq c \leq \ell$.

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The only-if part ($\Leftarrow$) is hard:

- There is not much we can assert for a general $n > 1$.
- Except that $n > 1$ must have a prime factor p that divides n.
- AKS modifications $\Rightarrow n = p^e$ where prime $p \mid n$ for some e.
- This is shown by a wonderful idea discovered by the AKS team.
- Once this is shown, power free check ensures $n = p$, a prime.

Introspective Relation

AKS polynomial identity checks involve double moduli:

$$(\boldsymbol{X} + \boldsymbol{c})^n \equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n,\ \boldsymbol{X}^k - 1}$$

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Rewriting with polynomial substitution, for a general ring $\mathcal{R}$:

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Denote $n \bowtie \mathrm{p}$: n is *introspective* to polynomial p, when:

$$\vdash n \bowtie \mathrm{p} \iff \mathrm{poly}\ \mathrm{p} \land 0 < k \land \mathrm{p}^n \equiv \mathrm{p}[\boldsymbol{X}^n] \pmod{\boldsymbol{X}^k - \mathbf{1}}$$

AKS Idea

The hard-part of AKS Main Theorem is proved by:

- Shifting the perspective from $\mathbb{Z}_n$ to $\mathbb{Z}_p$, where prime $p \mid n$.
- Showing that introspective relationship is preserved:

$$n \bowtie_{\mathbb{Z}_n} \boldsymbol{X} + \boldsymbol{c} \quad \Rightarrow \quad n \bowtie_{\mathbb{Z}_p} \boldsymbol{X} + \boldsymbol{c}$$

- Freshman-Fermat provides for free: $p \bowtie_{\mathbb{Z}_p} \boldsymbol{X} + \boldsymbol{c}$
- Construct sets based on these pair of introspective identities.
- Note that $\boldsymbol{X}^k - \mathbf{1} \in \mathcal{F}[X]$ has a monic irreducible factor $\boldsymbol{h}$.
- Construct finite sets based on parameter k and irreducible $\boldsymbol{h}$.
- By the choices of k and ℓ, squeeze out: $n = p^e$ for some e.

AKS Introspective Game

number $\bowtie$ polynomial

$(\boldsymbol{X} + \boldsymbol{c})^n \equiv \boldsymbol{X}^n + \boldsymbol{c} \pmod{\boldsymbol{X}^k - \mathbf{1}}$

$$n \bowtie (\boldsymbol{X} + \boldsymbol{c})$$
$$p \bowtie (\boldsymbol{X} + \boldsymbol{c})$$
$$0 < c \leq \ell$$

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number ⋈ polynomial

$$(\boldsymbol{X} + \boldsymbol{c})^n \equiv \boldsymbol{X}^n + \boldsymbol{c} \pmod{\boldsymbol{X}^k - \mathbf{1}}$$

$\mathcal{N}$: Numbers

$$\mathcal{N} = \{m \mid \gcd(m, k) = 1 \wedge \forall\, 0 < c \le \ell, m \bowtie \boldsymbol{X} + \boldsymbol{c}\}$$

$$n \in \mathcal{N}, p \in \mathcal{N}, 1 \in \mathcal{N}.$$

$n \bowtie (\boldsymbol{X} + \boldsymbol{c})$

$p \bowtie (\boldsymbol{X} + \boldsymbol{c})$

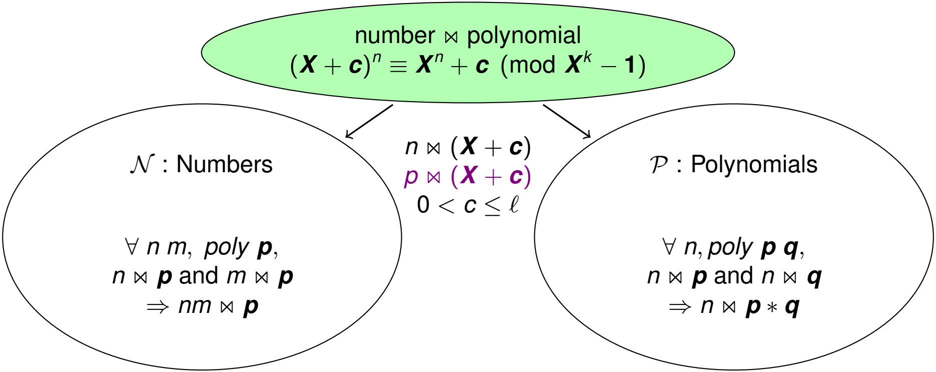
$0 < c \le \ell$

$\mathcal{P}$: Polynomials

$$\mathcal{P} = \{poly\ \boldsymbol{p} \mid \forall\, m \in \mathcal{N}, m \bowtie \boldsymbol{p}\}$$

$$\forall\, 0 < c \le \ell, \boldsymbol{X} + \boldsymbol{c} \in \mathcal{P},$$
$$\mathbf{1} \in \mathcal{P}.$$

AKS Introspective Game



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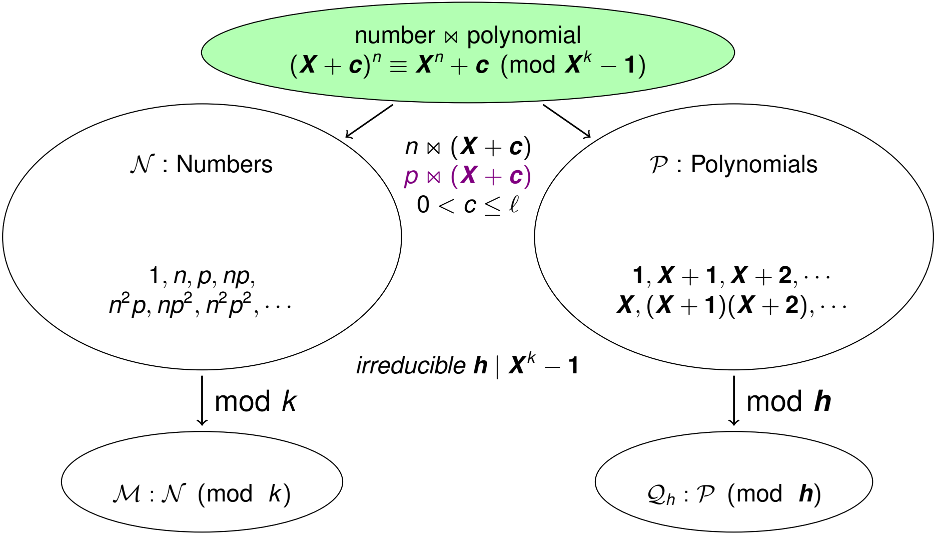

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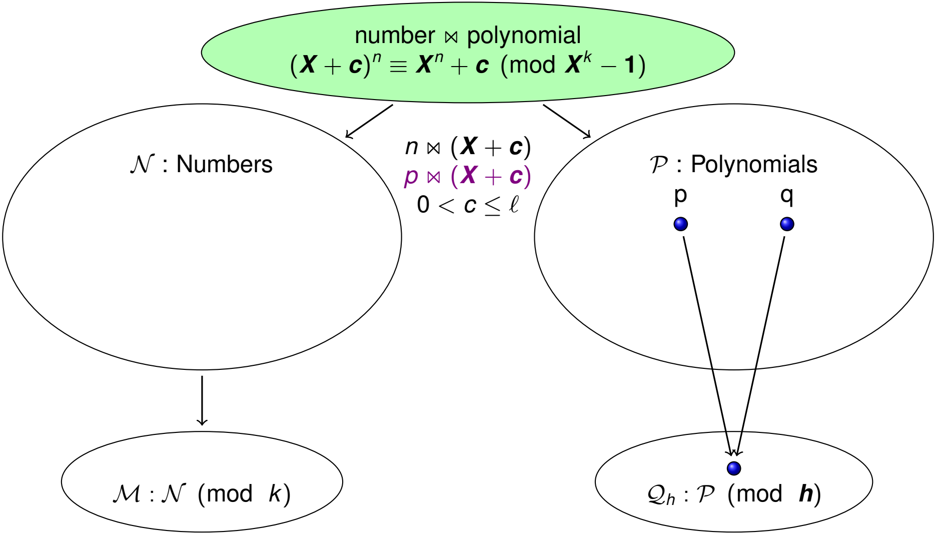

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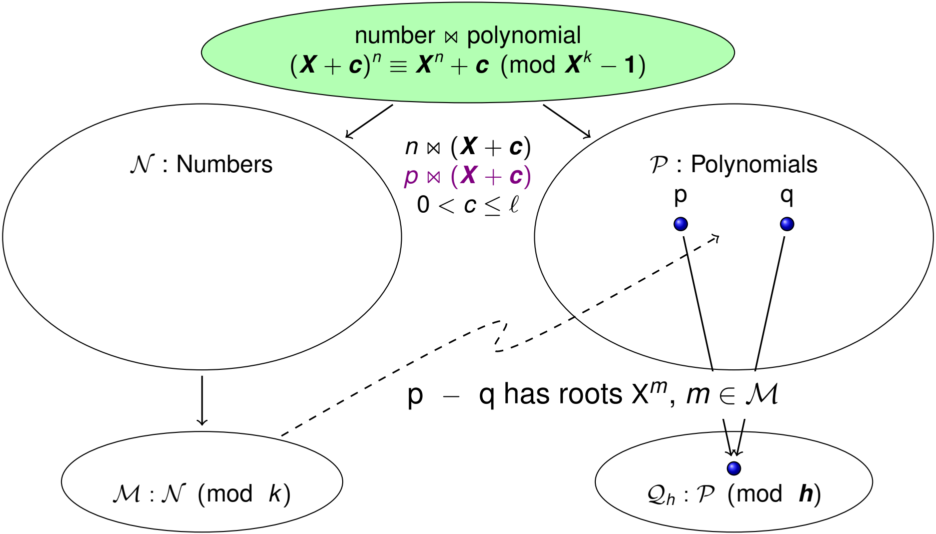

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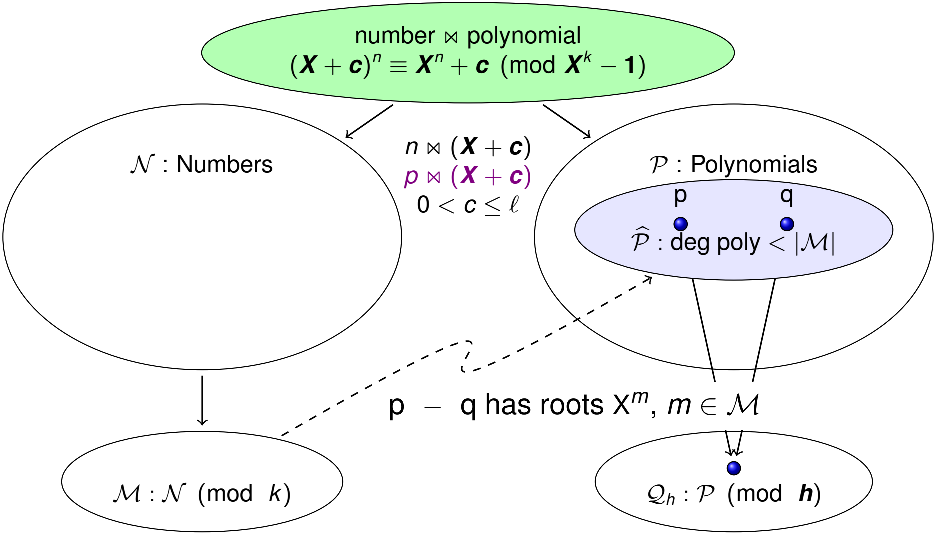

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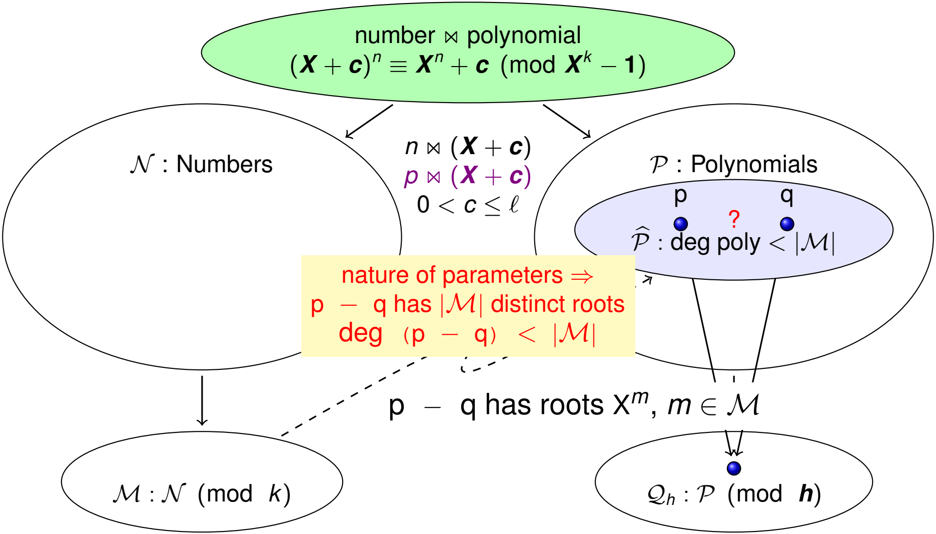

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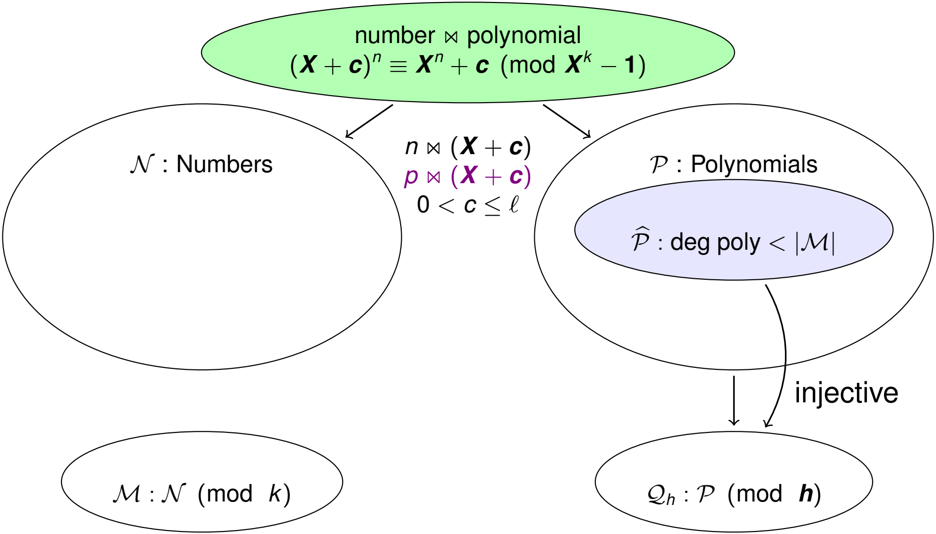

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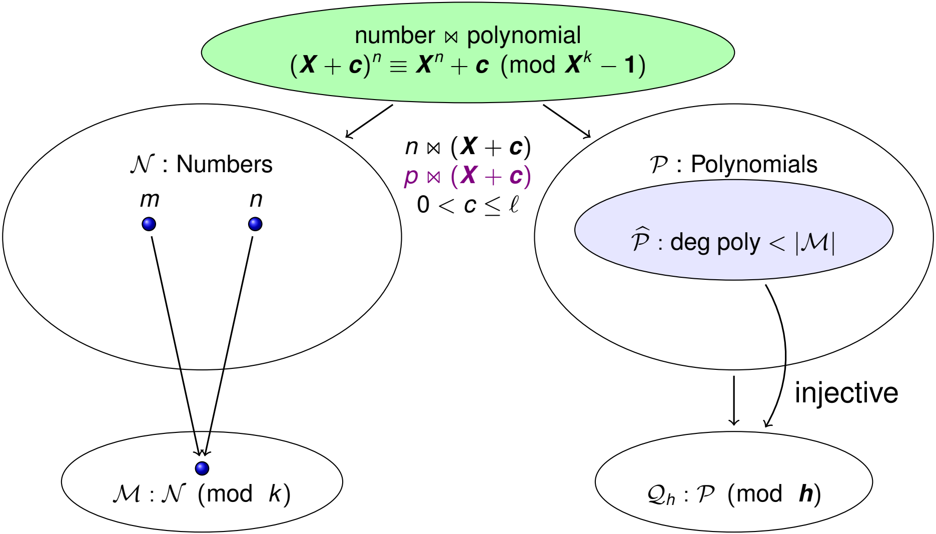

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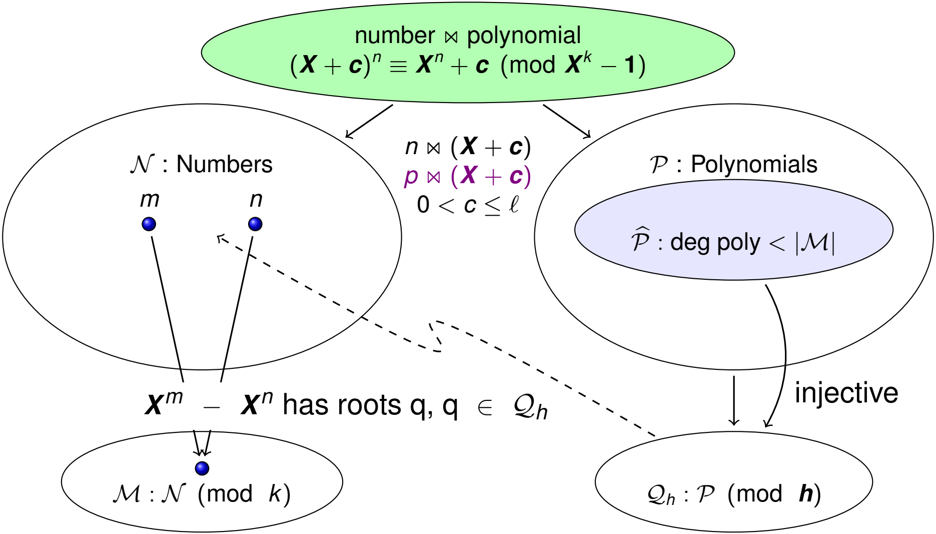

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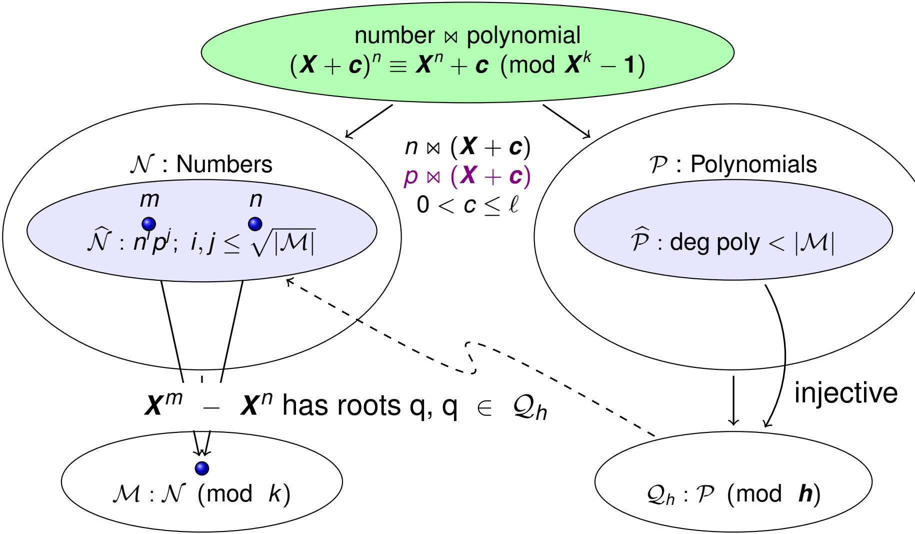

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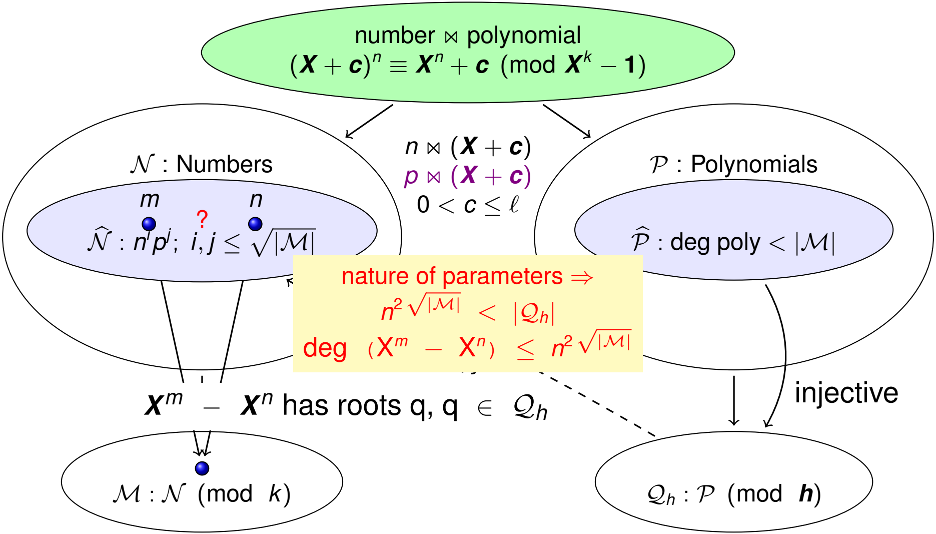

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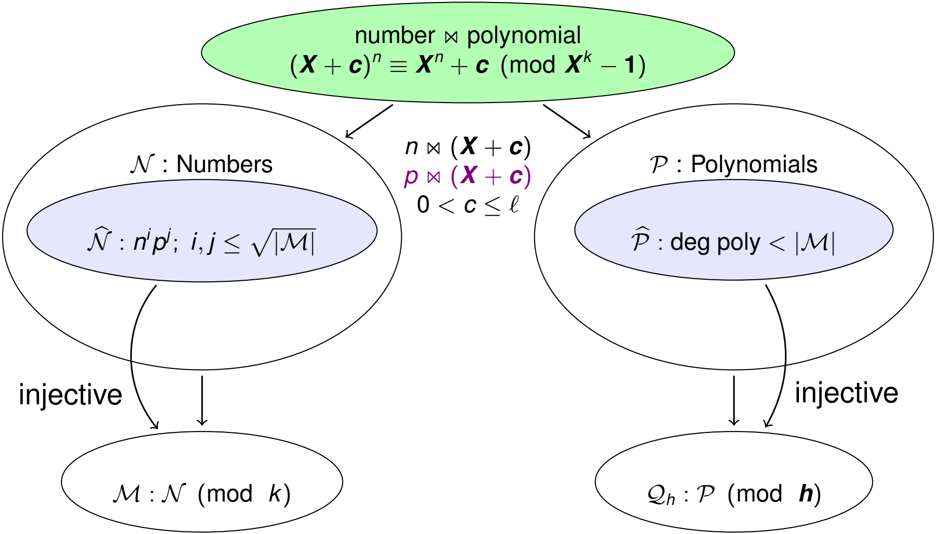

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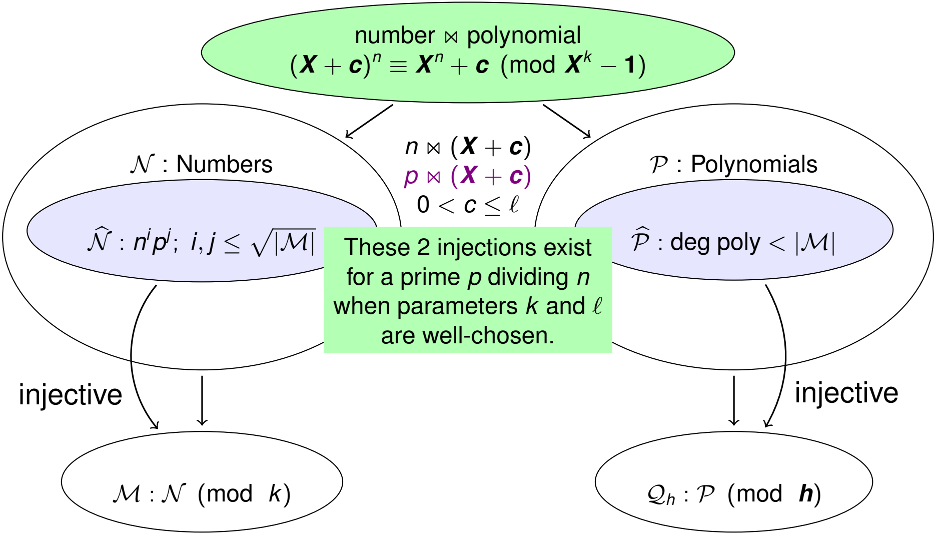

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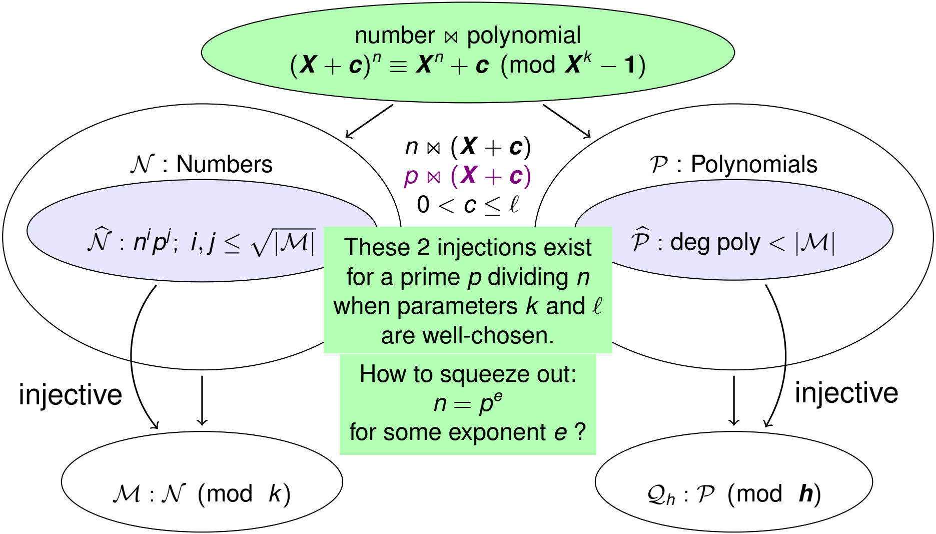

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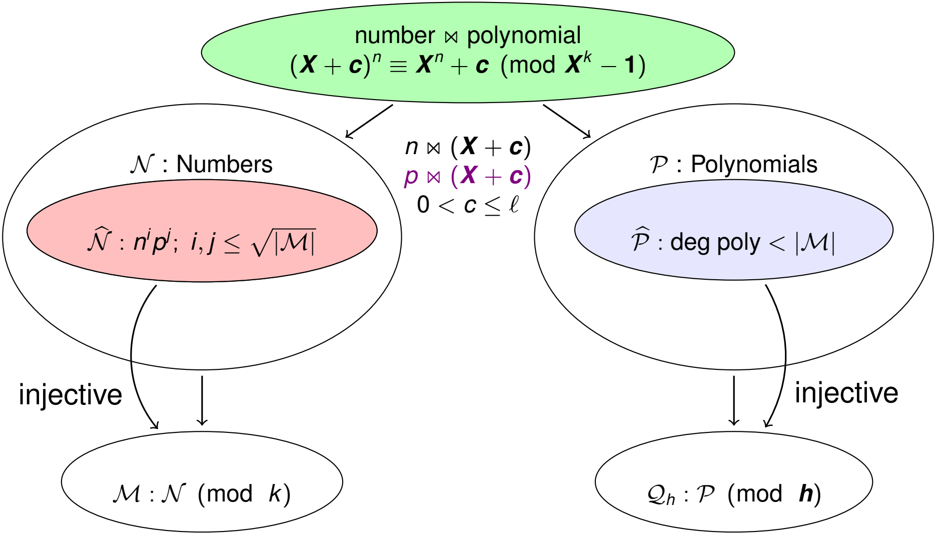

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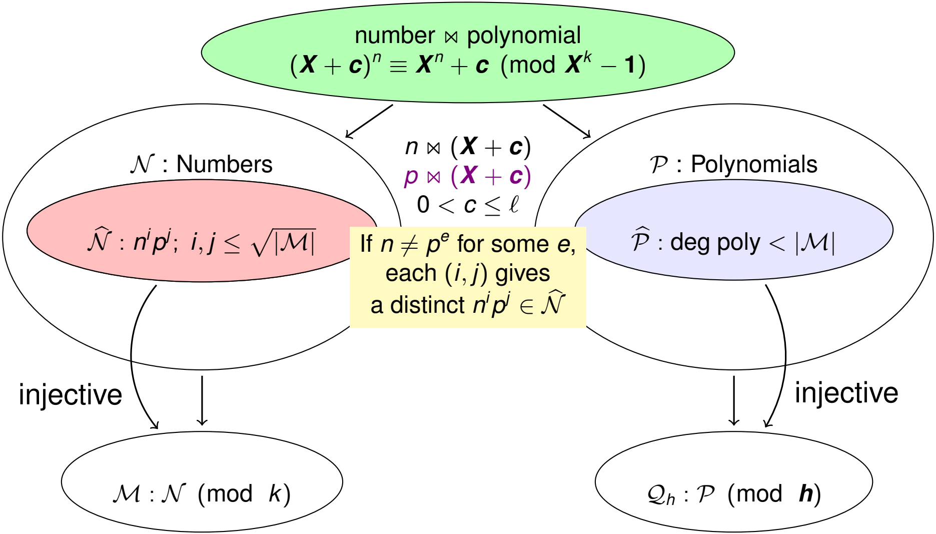

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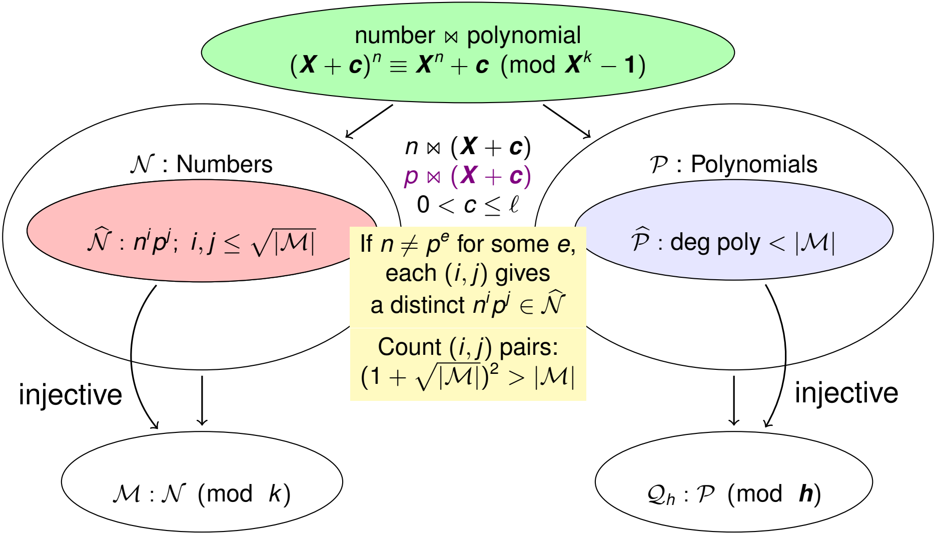

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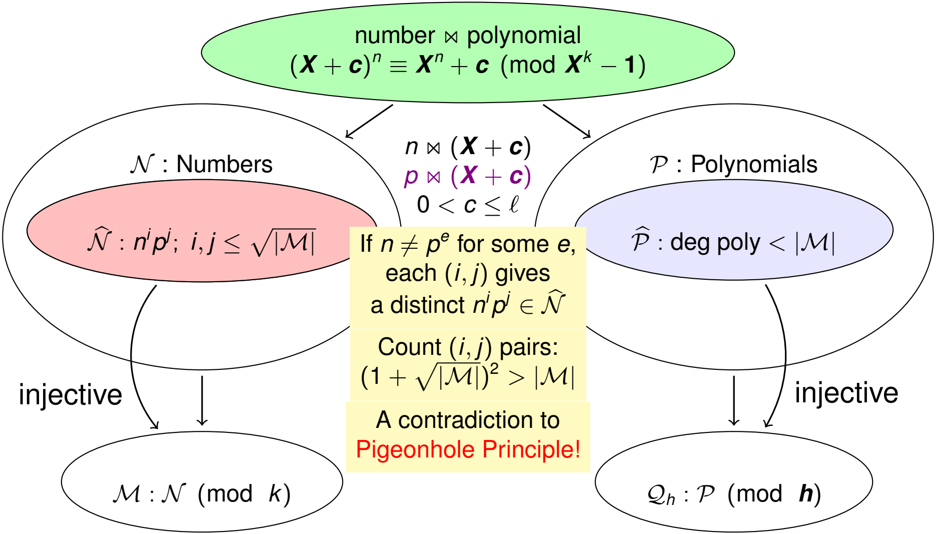

HOL Mechanisation

Straightforward Approach + Overloading:

```
monoid = <| carrier: 'a -> bool;
                op: 'a -> 'a -> 'a;
                id: 'a
          |>`;
```

Followed by:

```
val _ = overload_on ("*", ``g.op``);
val _ = overload_on ("#e", ``g.id``);
val _ = overload_on ("G", ``g.carrier``);
```

Some Line Counts

Underlying Algebra Library	49272
AKS specific scripts	21835

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AKS specific scripts	21835
Core HOL	150741