Formal Proof that PRIMES is in P

with a taste of HOL4

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Outline

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- By $x^2 - y^2 = (x - y)(x + y)$,
  $91 = (10 - 3)(10 + 3) = 7 \times 13$.
- n is prime iff it is not a difference of two non-consecutive squares.

Is 91 prime?

AKS method (August 2002)

- Check that the number n is power-free, *i.e.*, not square, cube, *etc*.
- Find a suitable parameter k, and compute ℓ based on k and n.
- GCD checks: if $\gcd(n, j) = 1$ for $j = 1 \ldots k$.
- Polynomial checks: if $(x + c)^n \equiv x^n + c \pmod{n, x^k - 1}$ for $c = 1 \ldots \ell$.

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  - For power-free 91, parameter $k = 37$, $\ell = 36$.
  - GCD checks: found $\gcd(91, 7) \neq 1$, so `COMPOSITE` 91.

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  - Even if this is missed, Polynomial checks give:

$$(x + 1)^{91} \not\equiv x^{91} + 1 \pmod{91, x^{37} - 1}$$

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- n is prime iff it passes all AKS checks.

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- note $10^2 = 100$ is nearest to 97, try $97 = 10^2 - y^2$, fail.
- fail $97 = 11^2 - y^2 = \cdots = 48^2 - y^2$ where $48 \approx \frac{97}{2}$, so $\texttt{PRIME}$ 97.

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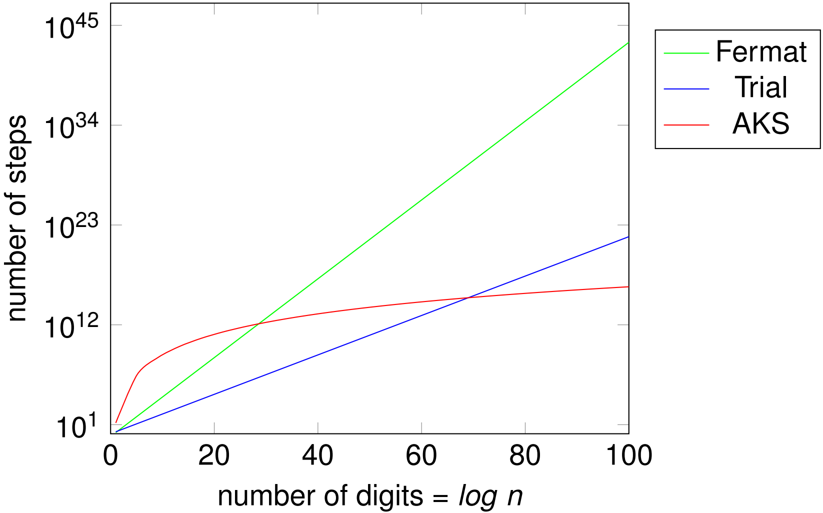
AKS method (August 2002)

- for power-free 97, parameter $k = 41, \ell = 37$.
- GCD checks: pass all $\gcd(97, j) = 1$, for $1 \leq j \leq 41$.
- Polynomial checks:
  - $(x + 1)^{97} \equiv x^{97} + 1 \pmod{97, x^{41} - 1}$ pass,
  - $(x + 2)^{97} \equiv x^{97} + 2 \pmod{97, x^{41} - 1}$ pass, $\cdots$, up to
  - $(x + 37)^{97} \equiv x^{97} + 37 \pmod{97, x^{41} - 1}$, all pass.
- hence 97 is $\texttt{PRIME}$.

Primality Tests Comparison



What is PRIMES in P?

PRIMES = the problem of Primality Testing

- Given an integer $n > 1$, determine if n is prime.

P = the class of Polynomial-time Algorithm

- When step count is bounded by a polynomial of input size ($\log n$).
- Such polynomial-time algorithms are deemed practical (or useful).

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Failure to show PRIMES $\in P$ promotes the belief: PRIMES $\notin P$ ...

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Methods

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Analysis

- size of $n =$ number of digits to represent n, measured by $\log n$.
- $O(n) = O(2^{\log n})$ is an exponential function of $(\log n)$.
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Title of AKS's 2002 paper: PRIMES is in P.
First known deterministic *polynomial-time* primality-testing algorithm.

The AKS Algorithm

6 steps in pseudo-code:

Input: integer $n > 1$.

1. If ($n = b^m$ for some base b with $m > 1$), return `COMPOSITE`.
2. Search for a prime k satisfying
   $\text{order}_k(n) \geq (2(\log n + 1))^2$.
3. For each ($j = 1$ to k) if ($j = n$) break,
   else if ($\gcd(n, j) \neq 1$), return `COMPOSITE`.
4. If ($k \geq n$), return `PRIME`.
5. For each ($c = 1$ to ℓ) where $\ell = 2\sqrt{k}(\log n + 1)$, if
   $(\boldsymbol{X} + \boldsymbol{c})^n \not\equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n,\ \boldsymbol{X}^k - 1}$, return
   `COMPOSITE`.
6. return `PRIME`.

The AKS Main Theorem

The AKS algorihtm works because it is based on:

Theorem (The AKS Main Theorem.)

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\begin{array}{l}
\vdash \text{prime } n \iff \\
\quad 1 < n \land \text{power\_free } n \land \\
\quad \exists k. \\
\qquad \text{prime } k \land (2 (\log n + 1))^2 \leq \text{order}_k(n) \land \\
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Proof.

ITP2015 `http://www.inf.kcl.ac.uk/staff/urbanc/itp-2015/`, a joint paper with my suprevisor, Michael Norrish. □

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- pre-conditions $\Rightarrow$ conclusion

A Mathematical Proof

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A Special Issue on Formal Proof
Notices of the American Mathematical Society, December 2008.
`http://www.ams.org/notices/200811/`

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I still have to formalize the AKS algorithm, and show it is indeed in P!

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HOL4 (sources on GitHub)

`http://hol-theorem-prover.org/`

`http://github.com/HOL-Theorem-Prover/HOL/`

AKS Source Repository

Source:
`http://bitbucket.org/jhlchan/hol/src/aks/theories`

- Helper Theories
  - AKSinteger — integer square-root and integer logarithm.
  - AKSpoly — polynomial evaluation by polynomial substitution.
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  - AKSintro — introspective relation essential to AKS proof.
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These are built upon other libraries:
algebraic structures, polynomials, finite fields, vector space, *etc*.

HOL Demo

First, set up the goal to be proved in HOL4 Proof Manager.

```
> g `1 + 1 = 2`;

val it =
   Proof manager status: 1 proof.
1. Incomplete goalstack:
     Initial goal:

     1 + 1 = 2

: proof
```

Then, apply one or more tactics to prove the goal:

```
> e (DECIDE_TAC);

OK..
val it = Initial goal proved.
|- 1 + 1 = 2: proof
```