# Formal Proof that PRIMES is in P

## with a taste of HOL4

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# Outline

1. Primality Testing
   - Examples
   - PRIMES in P
   - AKS Algorithm
   - AKS Theorem
2. Formalization
   - Formal Proof
   - Theorem Prover
   - Script Library
3. HOL4 Session
   - Live Demo

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- By $x^2 - y^2 = (x - y)(x + y)$,
  $91 = (10 - 3)(10 + 3) = 7 \times 13$.
- $n$ is prime iff it is not a difference of two non-consecutive squares.

# Is 91 prime?

AKS method (August 2002)

- Check that the number $n$ is power-free, *i.e.*, not square, cube, *etc*.
- Find a suitable parameter $k$, and compute $\ell$ based on $k$ and $n$.
- GCD checks: if $\gcd(n, j) = 1$ for $j = 1 \ldots k$.
- Polynomial checks: if $(x + c)^n \equiv x^n + c \pmod{n, x^k - 1}$ for $c = 1 \ldots \ell$.

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  - Again COMPOSITE 91.
- $n$ is prime iff it passes all AKS checks.

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- note $10^2 = 100$ is nearest to 97, try $97 = 10^2 - y^2$, fail.
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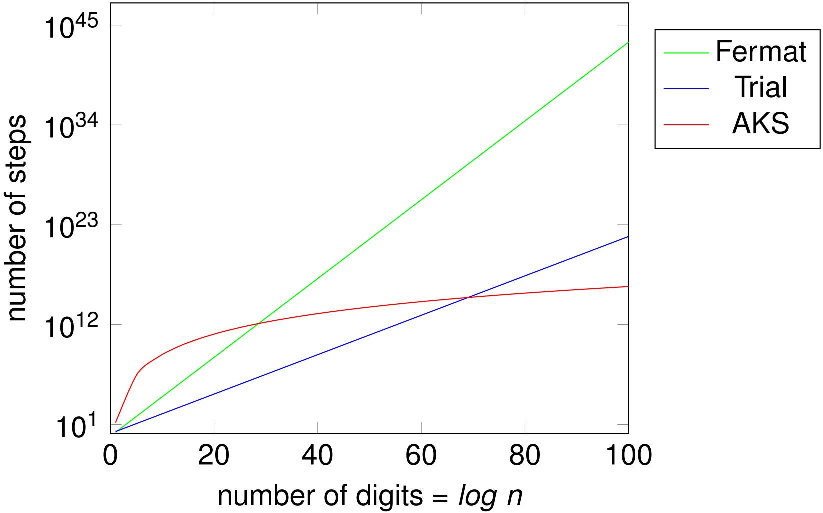
AKS method (August 2002)

- for power-free 97, parameter $k = 59, \ell = 48$.
- GCD checks: pass all $\gcd(97, j) = 1$, for $1 \le j \le 59$.
- Polynomial checks:
  - $(x + 1)^{97} \equiv x^{97} + 1 \pmod{97, x^{59} - 1}$ pass,
  - $(x + 2)^{97} \equiv x^{97} + 2 \pmod{97, x^{59} - 1}$ pass, $\cdots$, up to
  - $(x + 48)^{97} \equiv x^{97} + 48 \pmod{97, x^{59} - 1}$, all pass.
- hence 97 is $\texttt{PRIME}$.

# Primality Tests Comparison



# AKS paper

## PRIMES is in P

Manindra Agrawal Neeraj Kayal
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**Abstract**

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- `algebra/primality_original.pdf` (2002)
- `algebra/primality_v6.pdf` (revised, 2004)

# What is PRIMES in P?

PRIMES = the problem of Primality Testing

- Given an integer $n > 1$, determine if $n$ is prime.

P = the class of Polynomial-time Algorithm

- When step count is bounded by a polynomial of input size ($\log n$).
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Failure to show PRIMES $\in P$ promotes the belief: PRIMES $\notin P \ldots$

# AKS: PRIMES is in P

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Methods

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Analysis

- size of $n =$ number of digits to represent $n$, measured by $\log n$.
- $O(n) = O(2^{\log n})$ is an **exponential** function of $(\log n)$.
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Title of AKS's paper: PRIMES is in P.
First known deterministic *polynomial-time* primality-testing algorithm.

# The AKS Algorithm

Pseudo-code:

Input: an integer $n > 1$.

1. If ($n = b^m$ for some base $b$ with $m > 1$), return COMPOSITE.
2. For each ($k = 2$ to $m$) where $m = 2 + \lceil \log n \rceil * (\lceil \log n \rceil^4 / 2)$,
   - if ($\gcd(n, k) \neq 1$), if ($k = n$) return PRIME, else return COMPOSITE.
   - if ($\mathrm{order}_k(n) \geq \lceil \log n \rceil^2$), break (use this $k$ for next step).
3. For each ($c = 1$ to $\ell$) where $\ell = \sqrt{\varphi(k)} * \lceil \log n \rceil$,
   - if $(\boldsymbol{X} + \boldsymbol{c})^n \not\equiv (\boldsymbol{X}^n + \boldsymbol{c}) \pmod{n, \boldsymbol{X}^k - 1}$, return COMPOSITE.
4. return PRIME.

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4. return `PRIME`.

Implementation:
Check $n = b^m$, compute: $\gcd(n, k)$, $\text{order}_k(n)$, $\varphi(k)$, polynomials.

# The AKS Main Theorem

The AKS Algorithm is based on:

**Theorem (The AKS Main Theorem.)**

$$\begin{aligned}
\vdash\ & \text{prime}\ n \iff \\
&\quad 1 < n \wedge \text{power_free}\ n \wedge \\
&\quad \exists k. \\
&\qquad 1 < k \wedge (\log n + 1)^2 \le \text{order}_k(n) \wedge \\
&\qquad (\forall j.\ 0 < j \wedge j \le k \wedge j < n \Rightarrow \gcd(n, j) = 1) \wedge \\
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$$

## Proof.

To be written up in a joint paper with my suprevisor, Michael Norrish. □

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A Theorem

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A Special Issue on Formal Proof
Notices of the American Mathematical Society, December 2008.
`http://www.ams.org/notices/200811/`

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  - I have formalized the AKS algorithm, based on AKS Main Theorem.
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*Given enough eyeballs, all bugs are shallow*. (Linus's Law)

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I still have to formalize the AKS steps, to show algorithm is indeed in P!

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HOL4 (sources on GitHub)

`http://hol-theorem-prover.org/`

`http://github.com/HOL-Theorem-Prover/HOL/`

# AKS Source Repository

Source:

`http://bitbucket.org/jhlchan/hol/src/aks/theories`

- Helper Theories
  - AKSinteger — integer square-root and integer logarithm.
  - AKSorder — the existence of an AKS parameter related to order.
  - AKSpoly — polynomial evaluation by polynomial substitution.
- AKS Theories
  - AKSintro — introspective relation essential to AKS proof.
  - AKSshift — shifting introspective relation between Rings.
  - AKSsets — sets involved in AKS proof.
  - AKSmaps — mappings involved in AKS proof.
  - AKStheorem — the proof of AKS Main Theorem.
  - AKSimproved — the proof of termination of AKS algorithm.

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  - AKSintro — introspective relation essential to AKS proof.
  - AKSshift — shifting introspective relation between Rings.
  - AKSsets — sets involved in AKS proof.
  - AKSmaps — mappings involved in AKS proof.
  - AKStheorem — the proof of AKS Main Theorem.
  - AKSimproved — the proof of termination of AKS algorithm.

These are built upon other libraries:
algebraic structures, polynomials, finite fields, vector space, *etc*.

# HOL Demo

First, set up the goal to be proved in HOL4 Proof Manager.

```
> g `1 + 1 = 2`;

val it =
   Proof manager status: 1 proof.
1. Incomplete goalstack:
     Initial goal:

     1 + 1 = 2

: proof
```

Then execute by applying one or more tactics to prove the goal:

```
> e (DECIDE_TAC);

OK..
val it = Initial goal proved.
|- 1 + 1 = 2: proof
```