5+ methods for real analytic tetration

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June 28, 2010

Abstract

We feature different (mostly underexplored) methods for non-integer iteration that accumulated over the past decades and centuries. We show their relation if known and test them by application to exponentials. We prove that the matrix power iteration, when applied to a fixed point, is the (classic) regular iteration at that fixed point.

1 Introduction

Tetration in the sense of the fourth operation after addition, multiplication and exponentiation is widely discussed in lay mathematician communities. While multiplication is repeated addition and exponentiation is repeated multiplication, one would define tetration as repeated exponentiation:

$$b \cdot n := \underbrace{b + \cdots + b}_{n\times b} \qquad b^n = \underbrace{b \cdots b}_{n\times b} \qquad b\,[4]\,n = \underbrace{b^{b^{\cdot^{\cdot^{\cdot^{b}}}}}}_{n\times b}. \tag{1.1}$$

As exponentiation is not associative there would be other ways to bracket tetration and higher operations (for the use of arrows to indicate different bracketings see Bromer [Bro87]). In this article we are however only concerned with right bracketed tetration (1.1) and its extension to non-integer n.

The above formula (1.1) for tetration can be inductively written as

$$b\,[4]\,1 = b \qquad b\,[4](n+1) = b^{b\,[4]\,n}. \tag{1.2}$$

We want to keep this formula valid while extending from natural numbers n to non-integer n. Before we proceed with that let us see how much we can extend tetration towards negative integers. The formula (1.2) above is equivalent to $b\,[4](n-1) = \log_b(b\,[4]\,n)$. This gives then:

$$b\,[4]\,0 = 1 \qquad b\,[4](-1) = 0 \tag{1.3}$$

and $b\,[4](-2) = \log_b(0) = -\infty$. So naturally we would restrict the domain of any real-extended tetration to $(-2, \infty)$.

We assume that the reader is familiar with the behaviour of $x\,[4]\,n$ as $n \to \infty$. If not, it is a very illuminating exercise to plot the graphs for several n. A proof for the following proposition can be found in Knoebel's survey [Kno81].

Standard Knowledge 1 (Behaviour of $b\,[4]\,n$ for $n \to \infty$)**.** *The sequence $b_n = b\,[4]\,n$ converges for $e^{-e} \le b \le e^{1/e}$. The sequence b_{2n} is strictly decreasing, b_{2n+1} is strictly increasing and $b_{2n} > b_{2n+1}$ for $0 < b < 1$, both are convergent also for $0 < b < e^{-e}$. The sequence b_n is strictly increasing for $b > 1$.*

The oscillating behaviour of b_n and the monotone decrease of $x \mapsto b^x$ for $0 < b < 1$ is enough reason to exclude these bases b from our considerations. Indeed it can be shown (and be verified by the reader) that using the later described method of regular iteration (at the lower real fixed point) $b\,[4]\,x$ yields non-real complex values for most non-integer x.

However the critical point

$$\eta := e^{1/e} \approx 1.44466786 \tag{1.4}$$

will divide most of the later featured methods. Some of them only work for $1 < b < \eta$ or others work only for $b > \eta$, see the table at the end of the article. This has to do with the dependence of some methods on fixed points $z_0 = b^{z_0}$, see knowledge 14.

As the base b occurs always as a constant in (1.2) we switch from considering the tetration *operation* to considering tetration *functions* $f(x) = b\,[4]\,x$. We give it the dedicated name *super-exponential* or *tetrational* (differing from "ultra power" in [Hoo06] or "generalized exponential" in [Wal91a]) following a more extendible pattern (e.g. pentational, hexational, super-polynomial, super-factorial).

Definition 1 (super-exponential, tetrational)**.** We call the function f on D *super-exponential to base* b if it satisfies

$$f(x+1) = b^{f(x)} \tag{1.5}$$

for all x such that $x, x+1 \in D$. If additionally $f(n) = b[4]n$ for the smallest integer $n \in D$ (and hence for all other integers in D) then we call f *tetrational to base* b.

Note that any function f is a super-exponential if D does not contain $x+1$ for each $x \in D$, though we discourage this use of the definition. So if not stated otherwise we assume that D is incrementally closed, i.e. $x+1 \in D$ for all $x \in D$.

We can make a tetrational g from the (injective) super-exponential f if $1 \in f(D)$ by an argument shift:

$$g(x) := f(x + f^{[-1]}(1)).$$

See [KT10] for examples of super-exponentials where this is *not* possible.

To summarize: the scope of our article is: real analytic tetrationals on $(-2, \infty)$ to base $b > 1$.

1.1 Extended iteration, superfunction, Abel function and iterative logarithm

Definition 2 ($f^{[n]}$,$f^{[-1]}$)**.** Let $f\colon D \to D$, for natural numbers n we define $f^{[n]}\colon D \to D$ as $f^{[0]}(x) := x$ and

$$f^{[n]}(x) := \underbrace{f(\dots f(x)\dots)}_{n\times f} = \underbrace{(f \circ \dots \circ f)}_{n\times f}(x).$$

It satisifies $f^{[1]} = f$ and $f^{[m+n]} = f^{[m]} \circ f^{[n]}$. If f is bijective with the inverse g we define $f^{[-n]} := g^{[n]}$, particularly $g = f^{[-1]}$.

In the following we want to extend the iteration number to real or complex values. We put it however so general that the iteration number can be any element of a (additively written) monoid $(X, 0, +)$ with neutral element 0 and a distinct element $1 \in X$ (hence we always can assume $\mathbb{N}_0 \subseteq X$). For our purposes of $f = \exp_b$ you can consider X being the real line without $(-\infty, -2]$ or the complex plane without $(-\infty, -2]$.

Definition 3 (extended iteration, X-iteration, continuous iteration)**.** Let $f(D) \subseteq D$, a map $t \mapsto \beta^t$ that assigns each iteration number $t \in X$ a map $\beta^t\colon D \to D$ is called an X*-iteration* or *extended iteration* of f if it satisfies:

$$\beta^1 = f \qquad\qquad \beta^{s+t} = \beta^s \circ \beta^t \tag{1.6}$$

for all $s, t \in X$.

In the literature instead "iteration semigroup" as well as "continuous iteration" is used. The first is of rather cumbersome use (iteration semigroup of f over X) and sounds somewhat outdated while the latter is subject to the misunderstanding β^t being continuous. It is however synonymous for $\mathbb{R}$-iteration as $\mathbb{R}$ it also called continuum.

We see that $\beta^n = f^{[n]}$ for $n \in \mathbb{N}$.

In addition to *extended iteration* we will introduce the related terms *Abel function*, as well as *superfunction*. Roughly relation of the terms extended iteration/superfunction/Abel function is similar to the relation of power/exponential/logarithm.

Definition 4 ((intialized) superfunction, base function, super-exponential, tetrational)**.** Let $f(D) \subseteq D$, a map $u \mapsto \sigma_u$ that assigns to each initial value $u \in D$ a map $\sigma_u \colon X \to D$ is called a *u-initialized superfunction* of f on X if it satisfies:

$$\sigma_u(0) = u \qquad\qquad \sigma_u(t+1) = f(\sigma_u(t)) \tag{1.7}$$

for all $t \in X$ and all $u \in D$. The σ_u are called the *superfunctions* of f. We normally identify all superfunctions that only differ in an argument shift $\sigma_u \cong x \mapsto \sigma_v(x+c)$. f is called the *base function* of any σ_u. A superfunction of the exponential $\exp_b$ is called *super-exponential* to base b. Sometimes we give the dedicated name *tetrational* to base b to a superfunction σ_1 of $\exp_b$, then $\sigma_1(n) = \exp_b^{[n]}(1)$ is the power tower containing n instances of b.

It can be interpreted as the t-times application of f on u, particularly $\sigma_u(n) = f^{[n]}(u)$ for $n \in \mathbb{N}_0$ for any initialized superfunction σ.

The assignment of a superfunction is not unique. If σ is a superfunction then also $z \mapsto \sigma(z+c)$ is a superfunction of f. Sloppyly we identify superfunctions which are obtained by argument shifts. However there is a more severe non-uniqueness: If σ is a superfunction then also $z \mapsto \sigma(z+\theta(z))$ is a super-function of f for each 1-periodic θ.

In a corresponding way define the

Definition 5 ((initialized) Abel function, super-logarithm)**.** Let $f(D) \subseteq D$, a map $u \mapsto \alpha_u$ that assigns each initial value u a map $\alpha_u \colon D \to X$ is called *u-initialized Abel function* of f if it satisfies the Abel equation (right side) and the initial condition (left side) in

$$\alpha_u(u) = 0 \qquad\qquad \alpha_u(f(x)) = \alpha_u(x) + 1 \tag{1.8}$$

for all $x \in D$ such that $f(x) \in D$ and all $u \in D$. Each single α_u is just called an *Abel function* of f. We normally identify Abel functions that only differ by a constant $\alpha_u \cong \alpha_v + c$. We call an Abel function α_{-1} of $x \mapsto b^x$ *super-logarithm* (although it is not a superfunction of the logarithm.)

$\alpha_u(x)$ can be regarded as the number of applications of f applied to u that are needed to reach x.

Similar to the superfunction also the Abel function is not unique. If α is an Abel function then $z \mapsto \alpha(z)+c$ — or more generally $z \mapsto \alpha(z) + \theta(\alpha(z))$ for any 1-periodic θ — is also an Abel function of f. We usually identify two Abel functions that differ merely in a constant.

There is no such ambiguity in assigning the basefunction to an superfunction or Abel function. Each bijective superfunction σ has exactly one base function f and each bijective Abel function α has exactly one base function f. They are given by:

$$f(x) = \sigma(1 + \sigma^{[-1]}(x)) f(x) = \alpha^{-1}(1 + \alpha(x)).$$

Basic examples:

- $\beta^t(x) = x + bt$ is an $\mathbb{C}$-iteration of $f(x) = x + b$.
- $\beta^t(x) = b^t x$ is an $\mathbb{R}$-iteration of $f(x) = bx$.
- $\sigma(x) = bx$ is a 0-initialized superfunction of $f(x) = b + x$.
- $\sigma(x) = b^x$ is a 1-initialized superfunction of $f(x) = bx$.
- $\alpha(x) = x/b$ is a 0-initialized Abel function of $f(x) = x + b$.

Table of superfunctions

- $\alpha(x) = \log_b(x)$ is a 1-initialized Abel function of $f(x) = bx$.

Few examples of the superfunctions and the Abel functions are shown in Table 1.

BEGIN TODO: tidy up

Four realizations of tetration for various b are plotted in figure 1 for real values of the argument. As for the complex values, these tetrations are holomorphic at least in the domain $\{z \in \mathbb{C} : \Re(z) > -2\}$.

In generalization of the concept of a group acting on a set we could say β is an action of the semigroup X on D with $\beta^1 = f$. One only removes the demand of having inverses.

END TODO

The above 3 concepts extended iteration, initialized superfunction and Abel function are connected in the following way. From each image-bijective TODOwhats that? initialized superfunction $\sigma\colon D \to (X \leftrightarrow D)$ of f we obtain an image-bijective initialized Abel function $\alpha = A(\sigma)\colon D \to (D \leftrightarrow X)$ of f by taking the inverse:

$$A(\sigma) := \alpha \qquad \alpha_u = \sigma_u^{[-1]}.$$

From each image-bijective initialized Abel function $\alpha\colon D \to (D \leftrightarrow X)$ of f we obtain an X-iteration $B(\alpha)\colon X \to (D \leftrightarrow D)$ of f by

$$B(\alpha) := \beta \qquad \beta^t(z) := \alpha_u^{[-1]}(t + \alpha_u(z))$$

which is independendant on u. It is also bijective in t for fixed z.

From each such X-iteration $\beta\colon X \to (D \leftrightarrow D)$ of f we obtain an image-bijective initialized superfunction $\sigma\colon D \to (X \leftrightarrow D)$

$$S(\beta) := \sigma \qquad \sigma_u(t) := \beta^t(u)$$

for every $u \in D$. This last initialized superfunction is equal to our initial initialized superfunction σ:

$$(SBA(\sigma))_u(t) = \beta^t(u) = \alpha_u^{[-1]}(t + \alpha_u(u)) = \sigma_u(t),$$

We get the following further 3 identities

$$SBA(\sigma)_u = \sigma_u \qquad ASB(\alpha)_u = \alpha_u \qquad BAS(\beta) = \beta$$

and see that these 3 concepts are interchangeable.

There is another related term "iterative logarithm" which was coined by Jabotinsky [Jab63] as it is similar to the behaviour of the natural logarithm in several aspects (differently from the Abel function).

Definition 6 (iterative logarithm, Julia equation)**.** Let $f(D) \subseteq D$ be differentiable. A function λ on D that satisfies the Julia equation

$$\lambda(f(x)) = f'(x) \cdot \lambda(x) \tag{1.9}$$

is called an *iterative logarithm* of f.

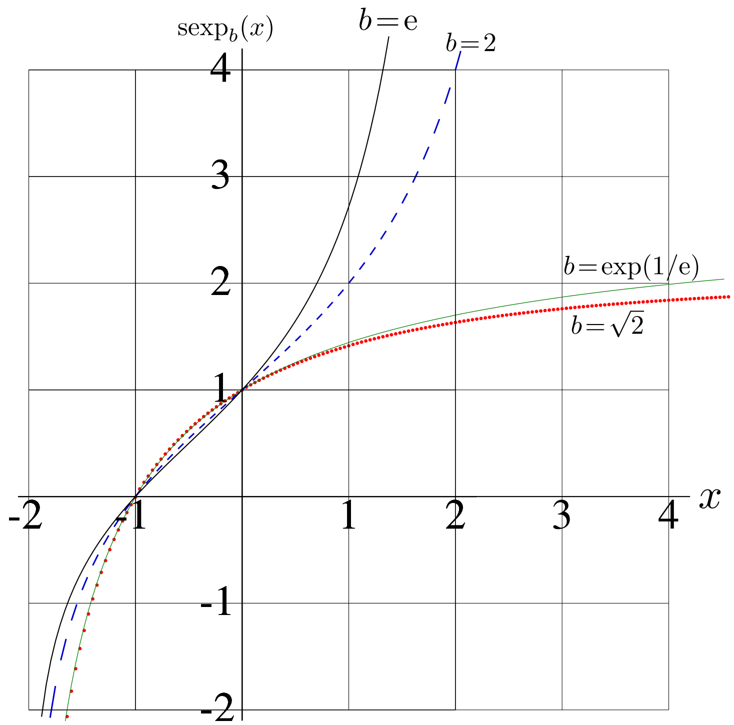

Figure 1: Examples of tetrationals $\mathrm{sexp}_b(x)$ at base $b=\mathrm{e}$ (thick solid), $b=2$ (dashed), $b=\exp(1/\mathrm{e})$ (thin solid) and $b=\sqrt{2}$ (dotted) versus real x.

You obtain this equation if you differentiate the Abel equation (1.8) and then set $\lambda(x) = 1/\alpha'_u(x)$.

Reversely if λ satisfies the Julia equation then $\gamma(z) = \int \frac{1}{\lambda(z)} dz$ satisfies $\gamma(f(z)) = \gamma(z) + c$ or equivalently $\frac{1}{c}\gamma(f(z)) = \frac{1}{c}\gamma(z) + 1$.

For an iterative logarithm λ

$$\alpha_u(x) = \frac{1}{c}\int_u^x \frac{1}{\lambda(\xi)} d\xi \quad \text{where} \quad c = \int_{f^{-1}(u)}^u \frac{1}{\lambda(\xi)} d\xi \tag{1.10}$$

is an initialized Abel function. The following left side is an iterative logarithm if β is an extended iteration of f, it satisfies the relation on the right side.

$$\lambda(x) := \left.\frac{\partial \beta^t(x)}{\partial t}\right|_{t=0} \qquad\qquad \lambda(\beta^t(x)) = t \cdot \lambda(\beta(x)). \tag{1.11}$$

Compare the relations

$$\left.\frac{\partial x^t}{\partial t}\right|_{t=0} = \ln(x) \qquad\qquad \ln(b^t) = t \cdot \ln(b)$$

for the natural logarithm.

1.2 Uniqueness

As the concepts extended iteration/superfunction/Abel function behave similar like power/exponential/logarithm, we first repeat uniqueness criterions for the latter concepts.

Standard Knowledge 2. *For each $b > 0$ there is exactly one real continuous function f satisfying*

$$f(1) = b \qquad\qquad f(s+t) = f(s)f(t)$$

and this solution is $f(x) = b^x$.

Complex powers/exponentials lack the above uniqueness. Thatswhy we define the following abbreviation:

Definition 7. $b^{z;k} := \exp((\log(b) + 2\pi i k)z)$ for $b \neq 0, w \in \mathbb{C}$, $k \in \mathbb{Z}$, where log is the standard logarithm determined by $-\pi < \Im(\log(z)) \leq \pi$. Omitting $;k$ means $k = 0$: $b^z := b^{z;0}$.

We see that the complex function $f(z) = b^{z;k}$ satisfies the equations of knowledge 2 for each $k \in \mathbb{Z}$. The next proposition shows that these are indeed all holomorphic solutions.

Standard Knowledge 3 ($b^{w;k}$)**.** *Let f be holomorphic in 0 satisfying $f(u+v) = f(u)f(v)$ for all $u, v, u+v$ in some vicinity of 0. Then f can be continued to an entire function. If $f(1) = b$ then $f(z) = b^{z;k}$ for some $k \in \mathbb{Z}$ and then $f'(0) = \log(b) + 2\pi i k$.*

2 $\mathfrak{R}$egular Iteration

Regular iteration is a well-investigated method due to the works of Kœnigs [Kœn84], Schröder [Sch70], Lévy [Lév28], Szekeres [Sze58], Écalle [Éca74], etc. Regular iteration with respect to a fixed point z_0 of f is an extended iteration β of f that is determined by being well-behaved at z_0. This means that $t \mapsto \beta^{t;k}$ being continuous and each $z \mapsto \beta^{t;k}(z)$ being at least asymptotically differentiable at z_0.

In particular, with regular iterations one can evaluate the superexponential at base $b \leq \exp 1/\mathrm{e}$. Four examples of the superexponentials for $b = \sqrt{2}$ are shown in figure 2

For real functions f one presumes $f'(z_0) > 0$ and distinguishes the parabolic case $f'(z_0) = 1$ from the other cases. (Complex functions are called parabolic at z_0 if $f'(z_0)$ is a root of unity.)

We start this section with regular iteration of formal powerseries with fixed point at 0, i.e. powerseries of the form $f(z) = \sum_{n=1}^{\infty} f_n z^n$. We denote the set of these powerseries with $\mathfrak{P}_0$. Later we consider the convergence radius of these powerseries and present equivalent limit formulas.

2.1 Formal Powerseries

Formal powerseries mimic the behaviour of powerseries developments of functions at 0, but have no need to converge. So for example $f(x) = x + 2^2x^2 + 3^3x^3 + \dots$ is a valid formal powerseries though it does not converge for any x.

For a formal powerseries f we usually denote the coefficient of x^n by $f_{:n}$. The formal powerseries f is completely determined by this sequence of its coefficients. So in this subsection we recall the formulas for the coefficients of the basic operations on powerseries, particularly obtaining a formula for composition.

We start with the well-known formulas for addition and multiplication.

$$\left(\sum_{n=0}^{\infty} f_{:n}x^n\right) + \left(\sum_{n=0}^{\infty} g_n x^n\right) = \sum_{n=0}^{\infty}(f_{:n} + g_{:n})x^n \qquad (f+g)_n = f_{:n} + g_{:n} \tag{2.1}$$

$$\left(\sum_{n_1=0}^{\infty} f_{:n_1}x^{n_1}\right)\left(\sum_{n_2=0}^{\infty} g_{:n_2}x^{n_2}\right) = \sum_{n,m=0}^{\infty} f_{:n_1}g_{:n_2}x^{n_1+n_2} \qquad (fg)_n = \sum_{n_1+n_2=n} f_{:n_1}g_{:n_2} \tag{2.2}$$

Division by f can be performed when $f_{:0} \neq 0$. We can derive the recursive formula of $g = 1/f$ by solving $fg = 1$ for g:

$$g_{:0} = \frac{1}{f_{:0}} \qquad g_{:n} = -\frac{1}{f_{:0}}\sum_{k=0}^{n-1} f_{:n-k}g_{:k}, \quad n \geq 1$$

When taking powers, note that $f^m{}_{:n}$ means the n-th coefficient of the m-th power of f, while $f_{:n}{}^m$ means the m-th power of the n-th coefficient of f. Deriving from (2.2) we get

$$f^m{}_{:n} = \sum_{n_1+\dots+n_m=n} f_{:n_1}\cdots f_{:n_m} = \sum_{\substack{m_1+2m_2+\dots+nm_n=n\\ m_0+\dots+m_n=m}} \frac{m!}{m_0!\cdots m_n!} f_{:0}{}^{m_0}\cdots f_{:n}{}^{m_n} \tag{2.3}$$

This enables a formula for composition

$$f(g(x)) = \sum_{m=0}^{\infty} f_{:m}g(x)^m \qquad (f\circ g)_{:n} = \sum_{m=0}^{\infty} f_{:m}g^m{}_n$$

which may however problematic as we dont know about the convergence of each coefficient. A minor constraint on g however makes the coefficients finite expressions.

If $g_{:0} = 0$ then $g^m{}_{:n} = 0$ for $n < m$. Hence

$$(f\circ g)_{:n} = \sum_{m=0}^{n} f_{:m}g^m{}_{:n}, \quad g_0 = 0. \tag{2.4}$$

Definition 8 ($\mathfrak{P}_0$)**.** That's why we introduce the symbol $\mathfrak{P}_0$ for the set of formal powerseries f with $f_0 = 0$.

It is closed under addition, multiplication and composition.

2.1.1 Non-Parabolic regular iteration of formal powerseries

We first notice that $(f\circ g)_1 = f_1g_1$

Standard Knowledge 4 (regular iteration on non-parabolic powerseries)**.** *Let P be the set of formal powerseries $f \in \mathfrak{P}_0$ with $0 < f_1$ and $f_1 \neq 1$. Then there exists exactly one $\mathbb{R}$-iteration β of f such that $\beta^t \in P$ for each $t \in \mathbb{C}$ and such that $t \mapsto \beta^t{}_1$ is continuous. It is called the* regular iteration *of f and written as $\beta = f^{\mathfrak{R}:}$.*

Proof. TODO. Uniqueness: consider the function $h(t) = {\beta^t}_1$. Then by definition 3 of the extended iteration: $h(1) = f_1$ and $h(s+t) = (\beta^s \circ \beta^t)_1 = h(s) \cdot h(t)$. By continuity $h(t) = {f_1}^t$. □

The regular iteration of f is recursively given by the following formula where $g := f^{\mathfrak{R}:t}$:

$$g_1 = {f_1}^t \qquad g_n = \frac{1}{{f_1}^n - f_1}\left(f_n {g_1}^n - g_1 f_n + \sum_{m=2}^{n-1} f_m {g^m}_n - g_m {f^m}_n \right). \tag{2.5}$$

This proposition can be generalized in the spirit of knowledge 3

Proposition 1 (Complex regular iteration on non-parabolic powerseries)**.** *Let P be the set of formal powerseries $f \in \mathfrak{P}_0$ with ${f_1}^n \neq f_1$ for all integer $n \geq 2$. If β is an $\mathbb{C}$-iteration of f such that $\beta^w \in P$ for each $w \in \mathbb{C}$ and such that $w \mapsto {\beta^w}_1$ is holomorphic in 0 then there exists a $k \in \mathbb{Z}$ such that ${\beta^w}_1 = {f_1}^{w;k}$.*

For each $k \in \mathbb{Z}$ there exists a unique $\mathbb{C}$-iteration β such that ${\beta^{w;k}}_1 = {f_1}^{w;k}$ which is called the regular iteration *of f. Each ${\beta^{w;k}}_n$ is a polynomial in $f_1^{w;k}$.*

Proof. TODO β is recursively given by (2.5) when replacing ${f_1}^t$ by ${f_1}^{w;k}$. □

Standard Knowledge 5 (principal Schröder powerseries)**.** *Let $f \in \mathfrak{P}_0$ be a formal powerseries with ${f_1}^n \neq f_1$ for all postive integers n, then there exists exactly one formal powerseries χ with $\chi_0 = 0$ and $\chi_1 = 1$ such that $\chi \circ f = f_1 \cdot \chi$. It is called the* principal Schröder powerseries *of f.*

The principial Schröder powerseries is recursively given by:

$$\chi_1 = 1 \qquad \chi_n = \frac{1}{f_1 - {f_1}^n} \sum_{m=1}^{n-1} \chi_m {f^m}_n. \tag{2.6}$$

The inverse $\chi^{[-1]}$ of the principal Schröder powerseries is recursively given by:

$${\chi^{[-1]}}_1 = 1 \qquad {\chi^{[-1]}}_n = \frac{1}{{f_1}^n - f_1} \sum_{m=1}^{n-1} f_m {\left(\chi^{[-1]}\right)^m}_n \tag{2.7}$$

Standard Knowledge 6. *Let $f \in \mathfrak{P}_0$, let β be the regular iteration of f and let χ be the principal Schröder powerseries of f. Then*

$$\beta^{w;k}(u) = \chi^{[-1]}({f_1}^{w;k} \cdot \chi(u)) = z \tag{2.8}$$

$$\alpha_u(z) = \log_{f_1;k} \frac{\chi(z)}{\chi(u)} = w \tag{2.9}$$

$$\lambda_0 = 0, \quad \lambda_1 = 1 \tag{2.10}$$

$$\lambda_n = \frac{1}{{f_1}^n - f_1} \sum_{m=1}^{n-1} \lambda_m \left((n-m+1) f_{n-m+1} - {f^m}_n\right) \tag{2.11}$$

2.1.2 Parabolic iteration of powerseries

Definition 9. Let $\mathfrak{P}_{01,N}$ be the set of formal powerseries f with $f_N \neq 0$ and $f_n = \mathrm{id}_n$ for $n < N$. Let $\mathfrak{P}_{01}$ be the union of all $\mathfrak{P}_{01,N}$, $N \geq 2$.

We first notice that $(f \circ g)_N = f_N + g_N$ for $f, g \in \mathfrak{P}_{01,N}$.

Let us now consider the parabolic case:

Standard Knowledge 7 (regular iteration on parabolic powerseries)**.** *For each $f \in \mathfrak{P}_{01}$ there exists a unique $\mathbb{C}$-iteration β of f such that $\beta^t \in \mathfrak{P}_{01}$ for each $t \in \mathbb{C}$ and such that each $t \mapsto {\beta^t}_n$ is continuous. The functions $t \mapsto {\beta^t}_n$ are polynomials. β is called the* regular iteration *of f. It is given by*

$$ {\beta^t}_1 = 1 \qquad\qquad {\beta^t}_n = \sum_{m=0}^{n-1} \binom{t}{m} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} {f^{[k]}}_n \tag{2.12} $$

TODO reference to Jabotinsky

A direct recursive formula for the coefficients can be given but it is too cumbersome to derive it at this place. Instead we refer to the formula by Jabotinsky TODO.

TODO Abel function as the Inverse of the superfunction $z_0 + \chi^{-1}(b^z)$ is $\log_b(\chi(z - z_0))$ (non-parabolic).

2.1.3 The regular Abel function via the Julia equation and iterative logarithm

The regular Abel function has a singularity at the fixed point of development, so we can not give a pure powerseries development as for the extended iteration of f. But we can obtain something similar.

By differentiating the Abel equation we get $\alpha'(f(x))f'(x) = \alpha'(x)$. Though α' can not be a powerseries when α isn't, it appears that $j(x) = 1/\alpha'(x)$ can be presented as a powerseries.

It satisfies the Julia equation, see (1.9).

Standard Knowledge 8 (iterative logarithm powerseries)**.** *Let $f \in \mathfrak{P}_0$ such that either $f_1 = 1$ or f_1 not being a primitive root of unity, let $(f - \mathrm{id})_n = 0$ for $n < N$ and $(f - \mathrm{id})_N \neq 0$ then there is exactly one solution $\lambda \in \mathfrak{P}_0$ with $\lambda_n = 0$ for $n < N$ and $\lambda_N = 1$ that satisfies the Julia equation (1.9). It is called the* iterative logarithm powerseries *of f and we write $\lambda = \mathrm{logit}[f]$. In the parabolic case $N \geq 2$ it is given by*

$$ \lambda_n = \sum_{m=0}^{n-1} \frac{s(m,1)}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} {f^{[k]}}_n \tag{2.13} $$

where $s(m,k)$ is the stirling number of the first kind and in the non-parabolic case $N = 1$ it is recursively given by

$$ \lambda_1 = 1 \qquad\qquad \lambda_n = \frac{1}{f_1^n - f_1} \sum_{k=1}^{n-1} ((n+1-k) f_{n+1-k} - {f^k}_n) \lambda_k, \quad n \geq 2. \tag{2.14} $$

So generally the iterative logarithm powerseries is of the form:

$$ \lambda(z) = \sum_{n=N}^{\infty} \lambda_n z^n = z^N \sum_{n=0}^{\infty} \lambda_{n+N} z^n, \quad \lambda_N \neq 0 \tag{2.15} $$

Hence its reciprocal can be given as

$$ \frac{1}{\lambda(z)} = z^{-N} \frac{1}{\sum_{n=0}^{\infty} \lambda_{n+N} z^n} = \sum_{n=-N}^{\infty} h_n z^n \tag{2.16} $$

and integrated

$$ \int \frac{1}{\lambda(z)} dz = \underbrace{\sum_{n=-N}^{-2} h_n \frac{z^{n+1}}{n+1} + h_{-1} \log(z)}_{\tilde{\alpha}(z)} + \sum_{n=0}^{\infty} h_n \frac{z^{n+1}}{n+1}. \tag{2.17} $$

Once we know this representation we can directly compute the coefficients in $\alpha_{-N+1} z^{-N+1} + \cdots + \alpha_{-1} z^{-1} + \alpha_* \log(z)$.

Proposition 2. *Let λ be the iterative logarithm powerseries of $f \in \mathfrak{P}_0$. Let*

$$\gamma(z) = \int \frac{1}{\lambda(z)} dz$$

$$\alpha_u(z) = \frac{\gamma(z) - \gamma(u)}{c} \qquad\qquad c = \begin{cases} \log(f_1) & N = 1 \\ f_N & N \geq 2 \end{cases}$$

then α_u is the regular Abel function of f.

TODO In the non-parabolic case $N = 1$ we obtain $c = \gamma(f(u)) - \gamma(u) = \log(f_1)$ and in the parabolic case we obtain $c = f_N$.

TODO search better place for this proposition, which is needed in the application to exponentials

Proposition 3. $\left(g^{[-1]} \circ f \circ g\right)^{\mathfrak{R}:w} = g^{[-1]} \circ f^{\mathfrak{R}:w} \circ g$ *for any $f, g \in \mathfrak{P}_0$ such that the equation is defined.*

2.2 Limit Formulas

From the last formula (2.17) we can already derive a limit formula. Considering $\alpha(f^{[n]}(x)) = n + \alpha(x)$ and $\alpha(z) - \tilde{\alpha}(z) \to 0$ for $z \to 0$ we obtain for $0 < |f_1| < 1$ (attractive fixed point at 0)

$$\alpha(x) = \lim_{n\to\infty} \tilde{\alpha}(f^{[n]}(z)) - n \tag{2.18}$$

2.2.1 Non-Parabolic

Proposition 4 (Principal Schröder function)**.** *Let $f \in \mathfrak{P}_0$ with $0 < |f_1| < 1$ have positive convergence radius then the principal Schröder powerseries χ of f has positive convergence radius too and is equal to the limit*

$$\chi(x) = \lim_{n\to\infty} \frac{f^{[n]}(x)}{{f_1}^n}$$

for each x in the basin of attraction of f at the fixed point 0.

$$\chi^{[-1]}(x) = \lim_{n\to\infty} f^{[-n]}\left({f_1}^n \cdot x\right)$$

($f^{[-1]}$ is the local inverse of f at 0.)

Proposition 5 (Regular iteration)**.** *Let $f \in \mathfrak{P}_0$ with $0 < |f_1| < 1$ have positive convergence radius then each regular iterate β^t of f has positive radius of convergence and is equal to the limit*

$$\beta^{t;k}(u) = \lim_{n\to\infty} f^{[-n]}\left({f_1}^{t;k} \cdot f^{[n]}(u)\right) = v$$

$$\chi_u(v) = \lim_{n\to\infty} \frac{f^{[n]}(v)}{f^{[n]}(u)} = {f_1}^{t;k}$$

2.2.2 Parabolic

TODO Powerseries has usually 0 convergence radius, but asymptotic powerseries. How to compute the function with non-converging powerseries (Julia-equation). Jabotinksy (better after matrices).

$$\beta^t(u) = \lim_{n\to\infty} f^{[-n]}((1-t) \cdot f^{[n]}(u) + t \cdot f^{[n+1]}(u)) = v \tag{2.19}$$

$$\alpha_u(v) = \lim_{n\to\infty} \frac{f^{[n]}(v) - f^{[n]}(u)}{f^{[n+1]}(u) - f^{[n]}(u)} = t \tag{2.20}$$

$$\alpha(v) = \lim_{n\to\infty} \frac{f^{[n]}(v) - f^{[n]}(u)}{f_N(f^{[n]}(u))^N}, \quad N \geq 2 \tag{2.21}$$

where N is the smallest number where f deviates from id.

The powerseries development of the Abel function can be used despite non-convergence in the following way: $\tilde{\alpha} = \int \frac{dz}{\lambda(z)}$

$$\alpha(z) = \lim_{n\to\infty} \tilde{\alpha}(f^{[n]}(z)) - n \tag{2.22}$$

2.2.3 Non-Zero Fixed Point

The previous formulas all assume that the fixed point is at 0. If we have a function f with fixed point at z_0 then we just consider the by a translation conjugated function $h(x) = f(x + z_0) - z_0$. This function has the fixed point at 0, can be iterated there and it then will be conjugated back.

Definition 10. Let f be a function with fixed point at z_0, let $h(z) := f(z + z_0) - z_0$, we call

$$f^{\Re z_0:w}(z) = h^{\Re:w}(z - z_0) + z_0 \tag{2.23}$$

the regular iteration of f at z_0.

If we set $\tau_{z_0}(z) := z + z_0$ then we can write $h = \tau_{z_0}^{[-1]} \circ f \circ \tau_{z_0}$ and the above formula as

$$f^{\Re z_0:w} = \tau_{z_0} \circ h^{\Re:w} \circ \tau_{z_0}^{[-1]} \tag{2.24}$$

Accordingly we rewrite the limit formulas for this general case, noticing that $(g^{[-1]} \circ f \circ g)^{[n]} = g^{[-1]} \circ f^{[n]} \circ g$:

Proposition 6. *Let f be holomorphic at z_0 and $\lambda := f'(z_0)$ with $0 < |\lambda| < 1$, the regular iteration β and the regular Schröder functions χ_z of f at z_0 are given by the limit formulas*

$$\beta^{w;k}(z) = \lim_{n\to\infty} f^{[-n]}\left((1 - \lambda^{w;k}) \cdot z_0 + \lambda^{w;k} \cdot f^{[n]}(z)\right) = v \tag{2.25}$$

$$\chi_z(v) = \lim_{n\to\infty} \frac{f^{[n]}(v) - z_0}{f^{[n]}(z) - z_0} = \lambda^{w;k}. \tag{2.26}$$

Proposition 7 (Lévy formula, parabolic iteration)**.** *Let f be holomorphic at z_0 and $f'(z_0) = 1$, the regular iteration β and the regular initialized Abel function α of f at z_0 are given by the (from z_0 independent) limit formulas*

$$\beta^w(z) = \lim_{n\to\infty} f^{[-n]}((1 - w) \cdot f^{[n]}(z) + w \cdot f^{[n+1]}(z)) = v \tag{2.27}$$

$$\alpha_z(v) = \lim_{n\to\infty} \frac{f^{[n]}(v) - f^{[n]}(z)}{f^{[n+1]}(z) - f^{[n]}(z)} = w. \tag{2.28}$$

Be careful that the above limit may also exist for functions with $f'(z_0) \neq 1$ where it will not give the regular itertion/Abel function but under the conditions of proposition 6:

$$\lim_{n\to\infty} \frac{f^{[n]}(v) - f^{[n]}(z)}{f^{[n+1]}(z) - f^{[n]}(z)} = \left(w + \frac{2\pi i k}{\log \lambda}\right) \frac{1 - \lambda^{w;k}}{1 - \lambda} \tag{2.29}$$

2.3 Application to Exponentials

2.3.1 formal powerseries iterates

We conjugate $f(x) = b^x$ to

$$
\begin{aligned}
h_1(x) &= f(x+b^-) - b^- = b^-(\mathrm{e}^{\ln(b)x} - 1) = \frac{1}{\ln b} h(x \ln b) \\
h(x) &= \ln(b) b^- (e^x - 1) = \underbrace{\ln(b^-)}_{f'(b^-)=:a} (e^x - 1)
\end{aligned}
$$

which has the powerseries development

$$
h_0 = 0 \qquad\qquad h_n = \frac{a}{n!}, \quad n > 0
$$

Regular $\mathbb{C}$-iteration:

$$
\begin{aligned}
h^{\mathfrak{R}:t}(z) &= sz + \frac{\frac{1}{2}s^2 - \frac{1}{2}s}{a-1} z^2 + \frac{\frac{1}{6}as^3 - \frac{1}{2}as^2 + \frac{1}{3}s^3 + \frac{1}{3}as - \frac{1}{2}s^2 + \frac{1}{6}s}{a^3 - a^2 - a + 1} z^3 + \cdots, \quad s = a^t \\
f^{\mathfrak{R}b^-:t}(z) &= b^- + \frac{1}{\ln b} h^{\mathfrak{R}:t}((z - b^-)\ln b)
\end{aligned}
$$

α obtained through the Schröder equation:

$$
\begin{aligned}
\chi_h(z) &= z + \frac{\frac{1}{2}}{-a+1} z^2 + \frac{\frac{1}{3}a + \frac{1}{6}}{a^3 - a^2 - a + 1} z^3 + \frac{-\frac{1}{4}a^3 - \frac{5}{24}a^2 - \frac{1}{4}a - \frac{1}{24}}{a^6 - a^5 - a^4 + a^2 + a - 1} z^4 + \cdots \\
\alpha_{\mathfrak{R}}[h](z) &= \log_a(\chi(z)) \\
\alpha_{\mathfrak{R}b^-}[f](z) &= \alpha_{\mathfrak{R}}[h]((z - b^-)\ln b) = \log_a(\chi((z - b^-)\ln b))
\end{aligned}
$$

α obtained through the Julia equation:

$$
\alpha_{\mathfrak{R}}[h](z) \ln a = \log(z) + \frac{\frac{1}{2}}{-a+1} z + \frac{\frac{5}{24}a + \frac{1}{24}}{a^3 - a^2 - a + 1} z^2 + \frac{\frac{1}{8}a^3 + \frac{1}{24}a^2 + \frac{1}{12}a}{-a^6 + a^5 + a^4 - a^2 - a + 1} z^3 + \cdots
$$

$$
\alpha_{\mathfrak{R}}[h](z) = \lim_{n\to\infty} \log_a(h^{[n]}(z)) - n = \log_a \frac{h^{[n]}(z)}{a^n}
$$

TODO refer back to the particular methods TODO refer to appendix for overview of fixpoints

2.3.2 limit formulas

To get the regular tetrational we apply this to $f(x) = b^x = \exp_b(x)$ at its lower fixed point $b^- = \mathrm{sr}_+^{[-1]}(b)$. The derivative at the fixed point b^- is $\lambda = \exp_b'(b^-) = \ln(b^-)$ by knowledge 15.

Definition 11. For $1 < b \le \eta$ we define $\mathrm{sexp}_b^{\mathfrak{R}}(x)$ to be the regular superfunction σ_1 of $z \mapsto b^z$ at its lower real fixed point b^-.

$$
b^- = \exp(-W(-\ln b)) = \frac{W(-\ln b)}{-\ln b} \tag{2.30}
$$

$$
\mathrm{sexp}_b^{\mathfrak{R}}(w) = \lim_{n\to\infty} \log_b^{[n]}\left((1 - (\ln b^-)^w) \cdot b^- + (\ln b^-)^w \cdot \exp_b^{[n]}(1)\right) \tag{2.31}
$$

$$\operatorname{rslog}_b(x) = \lim_{n\to\infty} \log_{\ln(b^-)} \frac{\exp_b^{[n]}(x) - b^-}{\exp_b^{[n]}(1) - b^-} \tag{2.32}$$

The parabolic case occurs for $b = \mathrm{e}^{1/\mathrm{e}}$ and fixed point e. The Lévy formula (7) converges much too slow to be applicable.

Conjugating the fixed point e of $f(x) = \mathrm{e}^{x/\mathrm{e}}$ to 0, we obtain:

$$h_1(x) = f(x+\mathrm{e}) - \mathrm{e} = \mathrm{e}^{(x+\mathrm{e})/\mathrm{e}} - \mathrm{e} = \mathrm{e}(\mathrm{e}^{x/\mathrm{e}} - 1)$$

We can further simplify the conjugated function to

$$h(x) = \mathrm{e}^x - 1, h_1(x) = \mathrm{e}h(x/\mathrm{e})$$

while knowing by proposition 3 that ${h_1}^{\mathfrak{R}:w}(z) = \mathrm{e}h^{\mathfrak{R}:w}(z/\mathrm{e})$ and of course $f^{\mathfrak{Re}:w}(z) = {h_1}^{\mathfrak{R}:w}(z-\mathrm{e}) + \mathrm{e}$, together:

$$f^{\mathfrak{Re}:w}(z) = \mathrm{e}h^{\mathfrak{R}:w}\left(\frac{z}{\mathrm{e}} - 1\right) + \mathrm{e}$$

$$\lambda(x) = x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{90}x^5 + \frac{11}{4320}x^6 - \frac{1}{3360}x^7 + \dots$$

From this iterative logarithm one can get a description of the regular Abel function by $\alpha = \frac{1}{f_2}\int \frac{1}{\lambda}$.

$$\alpha'(x) = \frac{1}{\lambda(x)} = x^{-2} + \frac{1}{6}x^{-1} - \frac{1}{72} + \frac{1}{540}x + \frac{1}{5184}x^2 - \frac{71}{217728}x^3 + \dots$$

If we integrate this and divide by $f_2 = \frac{1}{2}$ to get α the term x^{-1} becomes $\ln(x)$ for $x > 0$:

$$\alpha(x) = \underbrace{-2x^{-1} + \frac{1}{3}\ln(x)}_{\tilde{\alpha}(x)} - \frac{1}{36}x + \frac{1}{540}x^2 + \frac{1}{7776}x^3 - \frac{71}{435456}x^4 + \dots \tag{2.33}$$

Instead we get good results with Écalle's formula (2.18). To use this formula we h

3 $\mathfrak{M}$atrix power iteration

The basic observation for this method is that the composition of powerseries can be accomplished by the multiplication of certain associated matrices, the so called Carleman matrices:

Definition 12. Let f be a formal powerseries. We define the Carleman matrix $\mathrm{C}[f]$ as the infinite matrix that has as m-th row the coefficients of the powerseries f^m. The columns and rows here start at index 0, respectively. Formally:

$$\mathrm{C}[f]_{m,n} = {f^m}_{:n} \qquad m, n \in \mathbb{N}_0 \tag{3.1}$$

Standard Knowledge 9. *For formal powerseries f and g with $g_0 = 0$ we have* $\mathrm{C}[f\circ g] = \mathrm{C}[f]\,\mathrm{C}[g]$

Proof. Look at formula for powerseries composition (2.4) and apply the formula for powerseries powers (2.3) while noticing that ${g^k}_i = 0$ for $i < k$ because $g_0 = 0$:

$$\begin{aligned}
{(f\circ g)^m}_{:n} &= \sum_{n_1+\dots+n_m=n} \left(\sum_{k=0}^{n_1} f_{:k}{g^k}_{:n_1}\right)\cdots\left(\sum_{k=0}^{n_m} f_{:k}{g^k}_{:n_m}\right) \\
&= \sum_{k=0}^{n}\left(\sum_{k_1+\dots+k_m=k} f_{:k_1}\cdots f_{:k_m}\right){g^k}_{:n} = \sum_{k=0}^{n} {f^m}_{:k}{g^k}_{:n}
\end{aligned}$$

$\square$

3.1 Holomorphic Functions on Matrices

The most obvious way to apply a holomorphic function $f(z) = f_{:0} + f_{:1}z + f_{:2}z^2 + \dots$ to a matrix A is just by $f(A) = f_{:0} + f_{:1}A + f_{:2}A^2 + \dots$. However this approach does not always work: If the eigenvalues of A are not completely contained in the disk of convergence then the above method will fail.

We start with simpler matrices of the form

$$A = \begin{pmatrix} \lambda & a_{1,2} & \dots & a_{1,N-1} & a_{1,N} \\ 0 & \lambda & \dots & a_{2,N-1} & a_{2,N} \\ & & \vdots & & \\ 0 & 0 & \dots & \lambda & a_{N-1,N} \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix} \tag{3.2}$$

because then the matrix $A - \lambda I$ is nilpotent, $(A - \lambda I)^N = 0$. This means we do not need to take infinite sums:

Standard Term 1. For matrices of the form (3.2) we define the application of the holomorphic function f with powerseries development $f(z) = \sum_{n=0}^{\infty} f[z_0]_{:n}(z - z_0)^n$ at z_0 as

$$f(A) := \sum_{n=0}^{N-1} f[z_0]_{:n}(A - z_0 I)^n.$$

To make the application of the function invariant to matrix similarity we extend the definition to arbitrary square matrices A as follows.

Standard Term 2. Let f be a holomorphic function on D and let A be $N \times N$ square matrix with eigenvalues $\lambda_i \in D$, $1 \le i \le l$, which have multiplicities $\mu_i \ge 1$, $\mu_1 + \dots + \mu_l = N$. Let $A = SJS^{-1}$ be the Jordan normal form of A with $J = J_1 \oplus \dots \oplus J_l$, where J_i is the $\mu_i \times \mu_i$ Jordan block corresponding to λ_i (having λ_i on the diagonal and 1 on the upper secondary diagonal if $\mu_i \ge 2$). We define

$$f(A) := S\left(f(J_1) \oplus \dots \oplus f(J_l)\right) S^{-1}. \tag{3.3}$$

This definition is independent on the particular Jordan decomposition.

Indeed this definition is invariant under matrix similarity:

Standard Knowledge 10. *Let f be holomorphic on D then*

$$f(L^{-1}AL) = L^{-1}f(A)L \tag{3.4}$$

for any $N \times N$ square matrix A that has all eigenvalues in D and any invertible $N \times N$ matrix L.

Our definition coincides with direct application of the powerseries to the matrix if it converges:

Standard Knowledge 11. *Let f be a holomorphic function on D with powerseries expansion $f(z) = \sum_{n=0}^{\infty} f[z_0]_{:n}(z - z_0)^n$ at $z_0 \in D$. Let A be a complex square matrix with all eigenvalues contained in the convergence disk of f around z_0. Then*

$$f(A) = \sum_{n=0}^{\infty} f[z_0]_{:n}(A - z_0 I)^n.$$

Additionally the definition satsifies the following properties

Standard Knowledge 12. *Let f, g be holomorphic on D then*

$$(fg)(A) = f(A)g(A) \tag{3.5}$$

for every square matrix A that has all eigenvalues in D.

For proofs of these propositions see e.g. [HJ91].

With the previous establishments we can now easily define arbitrary powers of a a square matrix.

Definition 13 (matrix power)**.** Let A be a square matrix having only non-zero eigenvalues, we define *the matrix power* by $A^{w;k} := f_{w,k}(A)$ where $f_{w,k}(z) := z^{w;k}$.

Proposition 8. *$A^{1;k} = A$ and $A^{v+w;k} = A^{v;k}A^{w;k}$ for all complex v, w and integer k.*

Proof. Let $f_{w;k}(z) = z^{w;k} = \exp((\log(z) + 2\pi ik)w)$ then $f_{v+w;k}(z) = z^{v+w;k} = \exp((\log(z) + 2\pi ik)(v+w)) = z^{v;k}z^{w;k} = f_{v;k}(z)f_{w;k}(z)$ and further $A^{v+w;k} = f_{v+w;k}(A) = (f_{v;k}f_{w;k})(A) = f_{v;k}(A)f_{w;k}(A) = A^{v;k}A^{w;k}$ by (3.5). □

3.2 Matrix Power Iteration as Extension of Regular Iteration

Definition 14. We call the powerseries $\beta^{w;k}$ that has the coefficients of row 1 of $\lim_{N\to\infty} \mathrm{C_N}[f]^{w;k}$ the matrix power iteration of the powerseries f

$$f^{\mathfrak{M}:w;k}{}_{:n} := \left(\lim_{N\to\infty} \mathrm{C_N}[f]^{w;k}\right)_{1,n}. \tag{3.6}$$

Proposition 9. *Let $f \in \mathfrak{P}_0$, set $\lambda := f_1$. If $\lambda^n \neq \lambda$ for all positive integer n or if $\lambda = 1$ then the matrix power iteration of f is the regular iteration of f. More exactly:*

$$\begin{aligned} f^{\mathfrak{R}:w;k} &= f^{\mathfrak{M}:w;k} && \text{for} \quad \lambda^n \neq \lambda, \forall n \geq 2 \\ f^{\mathfrak{R}:w} &= f^{\mathfrak{M}:w,0} && \text{for} \quad \lambda = 1 \end{aligned}$$

Proof. The Carleman matrix of f is upper triangular. For upper triangular matrices S and T we have $(TS)|_N = (T|_N)(S|_N)$, that's why we can calculate with the infinite upper triangular matrices as with finite matrices.

In view of knowledge 1 and 7 we first verify that the matrix power iteration β of f is an extended iteration which follows from $A = \mathrm{C}[f]$ satisfying proposition 8.

In the case $\lambda^n \neq \lambda$, $n \geq 2$ all the eigenvalues $\lambda^n|_{n\in\mathbb{N}}$ are different, so all Jordan blocks have size 1. Hence each entry of $\mathrm{C}[f]^{w;k}$ is a sum of products of $(\lambda^n)^{w,k}$, i.e. holomorpic in w and hence matches the uniqueness condition of knowledge 1 and so must be the regular iteration.

In the case $\lambda = 1$ we have only one Jordan block J in the Jordan decomposition of $\mathrm{C_N}[f]$. This means that the entries of

$$J^{w;0} = J^w = \sum_{n=0}^{N} \binom{w}{n} (J - I)^n$$

are polynomials in w and particularly continuous. This matches the uniqueness condition of knowledge 7 and so must be the regular iteration. □

TODO matrix power with matrix exponential and logarithm. Matrix logarithm corresponds to iterative logarithm?

TODO the matrix power iteration, if converging, depends continuously on the development point. Hence Karlin-McGregor, dependence on the development point when walking between two fixed points.

3.3 Application to Exponentials

TODO see also Aldrovandi's unpublished introduction

The development of the Carleman matrix of the exponential $x \mapsto b^x$ can be given directly because the series of a power $x \mapsto b^{mx} = e^{\ln(b)mx}$ is obtained from the exponential series via some multiplication of the argument.

$$\mathrm{C}[\exp_b]_{m,n} = \frac{m^n \ln(b)^n}{n!} \tag{3.7}$$

If we however want a from 0 different development point x_0, i.e. the powerseries development of $b^{x+x_0} - x_0 = b^{x_0}b^x - x_0$ is somewhat more tedious:

$$(b^{x_0}b^x - x_0)^m = \sum_{k=0}^{m} \binom{m}{k} b^{kx_0} \overbrace{\left(\sum_{n=0}^{\infty} \frac{(k\ln(b)x)^n}{n!}\right)}^{b^{kx}} (-x_0)^{m-k} \tag{3.8}$$

$$\mathrm{C}[x \mapsto b^{x+x_0} - x_0]_{m,n} = \frac{\ln(b)^n}{n!} \sum_{k=0}^{m} \binom{m}{k} k^n b^{kx_0} (-x_0)^{m-k} \tag{3.9}$$

3.4 Discussion

TODO its open whether $C[f]^t$ is indeed the Carleman matrix of its first line for $N \to \infty$. This seems not the case for $0 < b < 1$.

TODO its open whether the coefficients converge. TODO its open whether the resulting series is convergent.

4 The Newton and Lagrange formulae for the regular superfunction

We can apply Newton's generalized binomial formula, which is valid for $|x-1| < 1$:

$$x^w = \sum_{n=0}^{\infty} \binom{w}{n} (x-1)^n = \sum_{n=0}^{\infty} \binom{w}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} x^m$$

also to square matrices A that have all eigenvalues $|\lambda_i - 1| < 1$:

$$A^w = \sum_{n=0}^{\infty} \binom{w}{n} (A-I)^n = \sum_{n=0}^{\infty} \binom{w}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} A^m$$

via proposition 11. Particularly this is possible for truncations of the Carleman matrix $\mathrm{C}[f]$, and if $f_0 = 0$ also for the (infinite upper triangular) Carleman matrix. It has the eigenvalues ${f_1}^n$, $n \in \mathbb{N}_0$. So if $0 < f_1 \leq 1$ then all eigenvalues of $\mathrm{C}[f]$ have distance < 1 from 1. If we take the first row on both sides we get:

$$f^{\mathfrak{R}:w}(z) = \sum_{n=0}^{\infty} \binom{w}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} f^{[m]}(z). \tag{4.1}$$

This is a powerseries (coefficient) equality. In the case $f_1 < 1$ both sides converge for z in some vicinity of 0. This vicinity can be extended to the basin of attraction (of 0). TODO, TODO case $f_1 = 1$.

Proposition 10. *Let f be a function with fixed point z_0, set $\lambda := f'(z_0)$. If $0 < \lambda \leq 1$ then regular iteration applied to z_0 can be expressed with (4.1) for z in the basin of attraction of z_0.*

Interestingly the superfunction version of (4.1) with $\sigma := \mathrm{regSf}_{z_0}[f, z_0]$

$$\sigma(w) = \sum_{n=0}^{\infty} \binom{w}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} f^{[m]}(z_0) \tag{4.2}$$

can also be interpreted as interpolating the points $(m, f^{[m]}(z_0))$, $0 \le m \le N$ for $N \to \infty$ via Newton's equidistant forward interpolation

$$\sigma_N(w) = \sum_{n=0}^{N} \binom{w}{n} \Delta^n[\sigma](0) \qquad\qquad \Delta^n[\sigma](0) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} \sigma(m). \tag{4.3}$$

We obtain the $0 \mapsto z_0$ superfunction (4.2) as $\sigma = \lim_{N\to\infty} \sigma_N$, noticing that $\sigma(m) = f^{[m]}(\sigma(0)) = f^{[m]}(z_0)$.

As an application of (4.2) we get the following beautiful series expression for the regular tetrational setting $z_0 = 1$:

$$\operatorname{sexp}_b^{\mathfrak{R}}(w) = \sum_{n=0}^{\infty} \binom{w}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} \exp_b^{[m]}(1). \tag{4.4}$$

We can also obtain the interpolating polynomial σ_N via the equidistant Lagrange interpolation:

$$\sigma(w) = \lim_{N\to\infty} \binom{w}{N+1} \sum_{m=0}^{N} (-1)^{N-m} \binom{N}{m} \frac{N+1}{w-m} f^{[m]}(z_0) \tag{4.5}$$

Taking $z_0 = 1$ this gives us another limit formula for the regular tetrational:

$$\operatorname{sexp}_b^{\mathfrak{R}}(w) = \lim_{N\to\infty} \binom{w}{N+1} \sum_{m=0}^{N} (-1)^{N-m} \binom{N}{m} \frac{N+1}{w-m} \exp_b^{[m]}(1) \tag{4.6}$$

One can also derive variants of these formulae in the same way by replacing x^w by $f^{\mathfrak{R}:w}$ which reminds a bit of umbral calculus. For example the regular tetrational to some base $b \in (1, \eta]$ approximates the values at $w < 0$ better if one replaces w by $w+1$ and $\exp_b^{[m]}(1)$ by $\exp_b^{[m-1]}(1) = \exp_b^{[m]}(0)$:

$$\operatorname{sexp}_b^{\mathfrak{R}}(w) = \sum_{n=0}^{\infty} \binom{w+1}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} \exp_b^{[m]}(0) \tag{4.7}$$

or the Lagrange form

$$\operatorname{sexp}_b^{\mathfrak{R}}(w) = \lim_{N\to\infty} \binom{w+1}{N+1} \sum_{m=0}^{N} (-1)^{N-m} \binom{N}{m} \frac{N+1}{w+1-m} \exp_b^{[m]}(0). \tag{4.8}$$

5 Kneser's Super-Logarithm Construction

In Kneser's original article TODO he constructs a real analytic Abel function of $\exp(z)$. His method can be easily generalzied to arbitrary bases $b > \eta$. TODO refer to appendix The important elements of such a construction are fixed points. The fixed points p as solutions of equation $b^p = p$ are shown in figure 6; in the $\Re(p)$,$\Im(p)$ coordinates for various values of b; the same $\Re(p)$ and $\Im(p)$ as functions of b are shown also in the right hand side of figure 4.

5.1 Steps of Kneser's Construction

We start with the Schröder function χ at the primary fixed point $p = b[1]$ in the upper halfplane, let $\lambda = \log(p) = {\exp_b}'(p)$. We focus our attention to the base region G that is bounded by the vertical line connecting p and the real axis and its image under b^z. χ maps G biholomorphically to G'. From the Schröder function we obtain an Abel function $\alpha(z) = \log_\lambda(\chi(z))$ by choosing the simply connected domain of the logarithm such that $G' = \chi(G)$ is completely contained in the domain of $\log_\lambda$. The logarithm maps G' biholomorphically to G''. α maps G biholomorphically to G''.

The main problem with this Abel function is that it is not real on the real axis. Even worse it has singularities at the points $\exp_b{}^{[n]}(0)$, $n \ge 0$ on the real axis.

Continuing the Abel function yields a multivalued function as there is a singularity at p. While the inverse Abel function $\sigma = \alpha^{-1}$, i.e. the regular superfunction, is entire.

maps all integer shifts of G'' to the upper halfplane via $\sigma(z+1) = \exp_b(\sigma(z))$. We would expect that a *real analytic* superfunction maps the upper halfplane R to the upper halplane. So we have a look at the pre-image $P = \sigma^{-1}(R)$.

By the Riemann mapping theorem there is (exactly one) transformation h that maps the union $P = \cup_{k\in\mathbb{Z}} G'' + k$ to the upper halfplane with $h(\alpha(1)) = 0$. This transformation h satisfies $h(z+1) = h(z)+1$. Hence $\tilde{\alpha}(z) = h(\alpha(z))$ maps the upper halfplane to the upper halfplane and satisfies $\tilde{\alpha}(\exp_b(z)) = \tilde{\alpha}(z) + 1$.

5.2 comments

Proposition 11. *The inverse Schröder function of a holomorphic self-map $f\colon G \to G$ at (repelling) fixed points $p \in G$ with $|f'(p)| > 1$ is entire and omits p.*

Proof. Without restriction let $p = 0$. By proposition 5 the inverse principal Schröder function has non-zero convergence radius r at 0 and satisfies $\chi^{-1}(\lambda z) = f(\chi^{-1}(z))$, $\lambda = f'(0)$, inside $|z| < r$. It can analytically continued to the whole complex plane by the derived equation $\chi^{-1}(\lambda^n z) = f^{[n]}(\chi^{-1}(z))$.

Now consider the local inverse $g = f^{-1}$ at 0 (which exists because $f'(0) \neq 0$) on a vicinity $|z| < r$. $\chi^{-1}(z/\lambda) = g(\chi^{-1}(z))$ and hence $\chi^{-1}(z\lambda^{-n}) = g^{[n]}(\chi^{-1}(z))$ for $\left|\chi^{-1}(z)\right| < r$. Suppose there was a z with $\chi^{-1}(z) = 0$ then $\chi^{-1}(z\lambda^{-n}) = 0$. Which implies by the identity theorem that $\chi^{-1}(z) = p$ for all z which is not possible as $\chi'(0) = 1$. ☐

TODO: Kneser's construction does not work at secondary fixed points.

6 The $\mathfrak{I}$ntuitive Abel Function

One surprisingly simple approach emerges when solving the infinite linear equation system that is obtained from the Abel equation applied to formal powerseries. Though this infinite equation system has an infinity of solutions (for each analytic Abel function α of f the function $\alpha(z)+\theta(\alpha(z))$ is again an analytic Abel function of f for every 1-periodic analytic θ) we consider the *intuitive* solution of the infinite equation system to be the limit of the solutions of the square-truncated linear equation systems. To our knowledge the method was first described by Walker [Wal91b].

Writing the Abel equation $\alpha(f(x)) = \alpha(x) + 1$ as formal powerseries with formula (2.4) we have:

$$\sum_{m=0}^{\infty} \alpha_{:m} f^m{}_{:n} = \alpha_{:n} + \delta_n^0$$

where δ_n^m is the Kronecker delta which is 1 if $m = n$ and 0 otherwise.

This is a(n infinite) linear equation system in α_m. If we subtract α_n on every n-th line we get:

$$\sum_{m=1}^{\infty} \left(f^m{}_{:n} - \delta_n^m\right) \alpha_m = \delta_n^0$$

We omit column (and variable) for $m = 0$ because it consists of 0 only due to $f^0{}_{:n} = 1$ for $n = 0$ and $f^0{}_{:n} = 0$ for $n > 0$. This indeed reflects the indeterminism of $\alpha_{:0}$ for any Abel function.

Though this equation system has an infinity of solutions (for every solution α the powerseries $\alpha(z) + \theta(\alpha(z))$ is another solution for every 1-periodic powerseries θ) there is one suggestive/*intuitive* way to obtain a solution. That is considering the square-truncated equation system

$$\sum_{m=1}^{N}\left(f^{m}{}_{:n}-\delta_{n}^{m}\right)\alpha_{:m}^{(N)}=\delta_{n}^{0},\quad 0\le n\le N-1$$

with the solution vector $\alpha^{(N)}$ and letting $N\to\infty$, i.e. setting $\alpha_{:m}=\lim_{N\to\infty}\alpha_{:m}^{(N)}$ if existing for $m\ge 1$ and $\alpha_{:0}=-1$ (to assure that $\alpha(1)=0$).

Several questions are open about this approach:

1. For which coefficients f_n does $\alpha_m^{(N)}$ converge (does it for $f(x)=b^x$, $b>1$)?
2. In which case does α have non-zero convergence radius (does it for $f(x)=b^x$, $b>1$)?
3. Is the obtained Abel function $\alpha[f]$ independent on the development point of f? More precisely: is $\alpha[\tau_s^{-1}\circ f\circ\tau_s]=\alpha[f]\circ\tau_s$ for $\tau_s(x)=x+s$ (at least for the shift s being inside the convergence disk of α)?
4. For the linear function $f(x)=bx$ one would expect the Abel funcion to be $\log_b$ (see examples on page 4). However the described procedure is not applicable for $f_0=0$. Instead one would consider its shift conjugate $f_s(x)=b(x+s)-s$, can we indeed confirm that $\alpha[f_s](x)=\log_c(x+s)$ for every s (or at least for $s=1$)?

Numerically however this approach looks quite promising.

6.1 Application to the exponential

If we assume that point 3 is true and if we assume that the Abel function has only singularities at the primary fixed points $b[\pm 1]$ of b^x we have an assumption about the convergence radius r of $\operatorname{slog}^{\mathfrak{I}}(z+s)$, i.e. $r=|b[1]-s|$ and about the convergence radius of the inverse $\operatorname{sexp}^{\mathfrak{I}}(z)-s$ which would be $r=\left|-2-\operatorname{slog}^{\mathfrak{I}}(s)\right|$.

To continue the intuitive Abel function to the complex plane one needs to transport each point of the complex plane into the disc of convergence around the development point z_0 by means of using $z\mapsto\exp_b(z)$ or $z\mapsto\log_b(z)$.

If we want to compute values less/greater than $\Re(b[1])$ we would choose s less/greater than $\Re(b[1])$.

TODO Compare numerically with $\mathfrak{R}$ for $b\le\eta$, $b=\sqrt{2}$, $b=\eta$.

TODO Compare numerically with $\mathfrak{C}$ for $b>\eta$, $b=2$ or $b=e$.

TODO Compare numerically with $\mathfrak{M}$ for $1<b$. $b=\sqrt{2},\eta,2,e$

TODO Compare numerically at different development points $b=\sqrt{2}$, $z_0=0,3,5$.

7 The $\mathfrak{C}$auchy Integral Approach

We are interested in *holomorphic* superfunctions, so the idea is not far to consider the Cauchy integral formula:

Standard Knowledge 13 (Cauchy integral formula)**.** *If f is holomorphic on D and if A is an open disk or rectangle such that $A\cup\partial A\subset D$, then*

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_{\partial A}\frac{f(\zeta)}{\zeta-z}d\zeta \tag{7.1}$$

for each $z\in A$. And vice versa if f is continuous on ∂A then the function

$$F(z):=\frac{1}{2\pi\mathrm{i}}\int_{\partial A}\frac{f(\zeta)}{\zeta-z}d\zeta$$

defined on the interior of ∂A is holomorphic.

Let $\gamma_|$ be a vertical (long though finite) line through a fixed $x_0 \in \mathbb{R}$. If we choose the left boundary of the rectangle A to be $\gamma_L = \gamma_| - 1$ and the right boundary to be $\gamma_| + 1$ and if we compute the values on $\gamma_|$ by the Cauchy integral formula then we can recover the values on γ_R and γ_L by $\sigma(z+1) = f(\sigma(z))$ and $\sigma(z-1) = f^{-1}(\sigma(z))$.

For the values on the top and bottom boundary γ_T and γ_B of A we consider the following. The Abel function (i.e. the inverse of σ) is supposed to have a singularity at a (chosen) fixed point. In the case of a real analytic function with a non-real fixed point z_1 there must also be a singularity on the conjugated fixed point z_{-1}. This means it would make sense to impose $\lim_{y\to\pm\infty} \sigma(x + \mathrm{i}y) = z_{\pm 1}$ for $x \in (-1, 1)$.

So for the rectangle A being tall enough one could just approximate the values on γ_T and γ_B by z_1 and z_{-1} respectively.

In formulas, we first define the parts of the boundary such that the concatenation of the parts is a counterclockwise closed path.

$$\gamma_L(t) = x_0 - 1 - \mathrm{i}t \qquad \begin{aligned} \gamma_{T,h}(t) &= x_0 - t + \mathrm{i}h \\ \gamma_|(t) &= x_0 + \mathrm{i}t \\ \gamma_{B,h}(t) &= x_0 + t - \mathrm{i}h \end{aligned} \qquad \gamma_R(t) = x_0 + 1 + \mathrm{i}t$$

The Cauchy integrals of σ along γ_L and γ_R for $z = \mathrm{i}y$ ($z \in \gamma_|$) are

$$\begin{aligned} \int_{\gamma_R} \frac{\sigma(\zeta)}{\zeta - (x_0 + \mathrm{i}y)} \mathrm{d}\zeta &= +\mathrm{i} \int_{-h}^{h} \frac{\sigma(x_0 + 1 + \mathrm{i}t)}{1 + \mathrm{i}t - \mathrm{i}y} \mathrm{d}t = \mathrm{i} \int_{-h}^{h} \frac{f(\sigma(x_0 + \mathrm{i}t))}{1 + \mathrm{i}(t - y)} \mathrm{d}t \\ \int_{\gamma_L} \frac{\sigma(\zeta)}{\zeta - (x_0 + \mathrm{i}y)} \mathrm{d}\zeta &= -\mathrm{i} \int_{-h}^{h} \frac{\sigma(x_0 - 1 - \mathrm{i}t)}{-1 - \mathrm{i}t - \mathrm{i}y} \mathrm{d}t = \mathrm{i} \int_{-h}^{h} \frac{f^{-1}(\sigma(x_0 - \mathrm{i}t))}{1 + \mathrm{i}(t + y)} \mathrm{d}t. \end{aligned}$$

For the top and bottom boundary:

$$\begin{aligned} \lim_{h\to\infty} \int_{\gamma_{T,h}} \frac{\sigma(\zeta)}{\zeta - (x_0 + \mathrm{i}y)} \mathrm{d}\zeta &= \lim_{h\to\infty} -z_1 \int_{-1}^{1} \frac{\mathrm{d}t}{-t + \mathrm{i}(h - y)} = \lim_{h\to\infty} z_1 \underbrace{\int_{-1}^{1} \frac{\mathrm{d}t}{t - \mathrm{i}(h - y)}}_{\lambda(h-y)} \\ \lim_{h\to\infty} \int_{\gamma_{B,h}} \frac{\sigma(\zeta)}{\zeta - (x_0 + \mathrm{i}y)} \mathrm{d}\zeta &= \lim_{h\to\infty} z_{-1} \int_{-1}^{1} \frac{\mathrm{d}t}{t - \mathrm{i}(h + y)} = \lim_{h\to\infty} z_{-1} \lambda(h + y) \\ \lambda(d) = \log(1 - \mathrm{i}d) - \log(-1 - \mathrm{i}d) &= 2\mathrm{i}\left(\frac{\pi}{2} - \arctan(d)\right), \quad d > 0 \end{aligned}$$

We put them together with the Cauchy integral formula 7.1 to obtain a recurrence

$$\sigma(x_0 + \mathrm{i}y) = \lim_{h\to\infty} \frac{1}{2\pi} \int_{-h}^{h} \left(\frac{f(\sigma(x_0 + \mathrm{i}t))}{1 + \mathrm{i}(t - y)} + \frac{f^{-1}(\sigma(x_0 - \mathrm{i}t))}{1 + \mathrm{i}(t + y)} \right) \mathrm{d}t + \frac{z_1 \lambda(h - y) + z_{-1} \lambda(h + y)}{2\pi \mathrm{i}}$$

This roughly describes the iteration process we use. We approximate $h \to \infty$ by a big enough number and establish a grid on $\gamma_|$ of σ values. The integration is carried out numerically.

Without changing the obtained $\sigma(x_0 + \mathrm{i}y)$ we can replace x_0 by any other number, particularly $x_0 = 0$.

It is however not yet proven (for $f(x) = e^x$ and any other function) that this iteration process indeed converges. Not even that if it converges then the resulting function is a holomorphic superfunction. On the other hand the numerical results support all these 3 claims ((1.)convergence to a (2.)holomorphic (3.)superfunction).

7.1 Application to the exponential

This function is shown in figure 2.

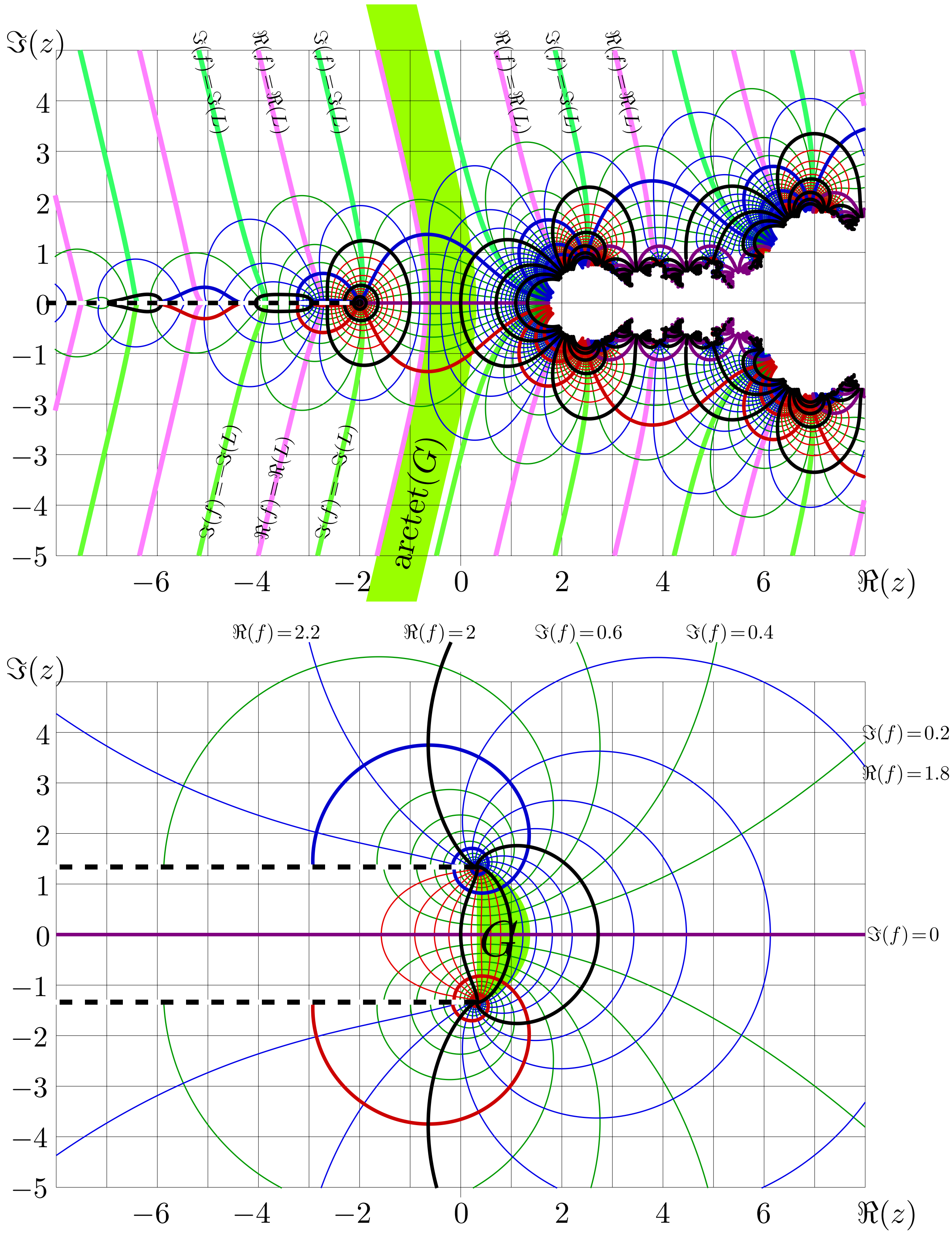

Figure 2: functions $f = \mathrm{tet}(z)$ and $f = \mathrm{arctet}(z)$ in the complex z-plane.

8 Reduction to known iteration

This method appears in different facets. The common principle is to reduce the iteration for high bases $b > \eta$ to the known (regular) iteration of some function, for example to an exponential with base $\leq \eta$.

8.1 Lévy's approach

To obtain an extended iteration of e^x Lévy proposes in his article [Lév27] the reduction to the iteration of $f(x) = e^x - 1$ in the following way: Let α be an Abel function of the decremented exponential $e^x - 1$ (i.e. the regular Abel function at fixed point 0), then the limit (which exists)

$$\operatorname{slog}_{e,z}(u) = \lim_{n\to\infty} \alpha(\exp^{[n]}(u)) - \alpha(\exp^{[n]}(z))$$

is an u-initialized Abel function of exp. This can be easier seen when inverting the above formula (using the regular iteration β of $e^x - 1$):

$$\operatorname{spow}_e^w(z) = u = \lim_{n\to\infty} \log^{[n]}(\beta^w(\exp^{[n]}(z)))$$

which we can more easily be seen to satisfy $\operatorname{spow}_e^{w_1+w_2}(z) = \operatorname{spow}_e^{w_1}(\operatorname{spow}_e^{w_2}(z))$ and $\operatorname{spow}^1(z) = \exp(z)$.

$$\operatorname{slog}_b(x) = \operatorname{slog}_a(\mu_{a,b}(x)) = \lim_{n\to\infty} \operatorname{slog}_a(\exp_b^{[n]}(x)) - n$$
$$\operatorname{slog}_{a,u}(x) = \operatorname{slog}_b(x) - \operatorname{slog}_b(u) = \lim_{n\to\infty} \operatorname{slog}_a(\exp_b^{[n]}(x)) - \operatorname{slog}_a(\exp_b^{[n]}(u))$$

8.2 Change of base

The base observation is that the following limit exists (and is real) for $1 < a < b$ and arguments $x \in \mathbb{R}$:

$$\mu_{a,b} := \lim_{n\to\infty} \log_a^{[n]} \circ \exp_b^{[n]} \tag{8.1}$$

For $1 < a < b$ we can make the following assertions:

1. If $b \leq \eta$
   (a) if $x \leq b^+$ then $\mu_{a,b}(x) = a^+$ (const.)
   (b) $\mu_{a,b}$ maps $(b^+, \infty) \mapsto (a^+, \infty)$. It is strictly increasing and infinitely differentiable on (b^+, ∞).
   (c) $\mu_{b,a}$ maps $(a^+, \infty) \mapsto (b^+, \infty)$ and is the inverse function of $\mu_{a,b}$.
2. If $b > \eta$
   (a) then $\mu_{a,b}$ is strictly increasing, unbounded above and infinitely differentiable [VdS88] TODO[covers that all bases?].
   (b) $\mu_{a,b}$ maps $(-\infty, \infty) \mapsto ((a,b)^+, \infty)$ where $(a,b)^+ := \log_a(\mu_{a,b}(0))$; $\mu_{a,b}(0) > 0$; $(a,b)^+ \geq a^+$ if $a \leq \eta$.
   (c) $\mu_{b,a}$ maps $((a,b)^+, \infty) \mapsto (-\infty, \infty)$ and is the inverse function of $\mu_{a,b}$.
3. $\mu_{a,b} \circ \mu_{b,c} = \mu_{a,c}$ for $1 < a < b < c$, $b > \eta$.

We call it base change because it obviously satisfies

$$\mu_{a,b} \circ \exp_b = \exp_a \circ \mu_{a,b}. \tag{8.2}$$

It is however not yet known whether this function is analytic. And despite the author's conjecture that it is nearly nowhere analytic, we mention this method here as we consider it an important idea.

If we got an Abel function slog_a of a^x (for example by regular iteration at the lower fixed point in the case $a \le \eta$) then we have also the Abel function slog_b of b^x, $b > a, \eta$, given by:

$$\mathrm{slog}_b = \mathrm{slog}_a \circ \mu_{a,b} \qquad \mathrm{sexp}_b = \mu_{b,a} \circ \mathrm{sexp}_a \tag{8.3}$$

8.3 Walker's approach

$$\begin{aligned}
\log_a \log_a \exp_b \exp_b(x) &= \frac{\ln b}{\ln a} x + \underbrace{\frac{\ln\ln b - \ln\ln a}{\ln a}}_{c_{a,b}} =: \tau(x) \\
\tau^{-1}(a^{\tau(x)}) &= \frac{\ln a e^{x \ln(b) + \ln\ln b - \ln\ln a} - \ln\ln b + \ln\ln a}{\ln b} \\
&= b^x + \frac{\ln\ln a - \ln\ln b}{\ln b} = b^x + c_{b,a} \\
\tau^{-1}(\log_a(\tau(x))) &= \log_b(x - c_{b,a}) =: \mathrm{logi}_{b,a}(x) \\
\log_a^{[n]} \circ \exp_b^{[n]} &= \tau \circ \mathrm{logi}_{b,a}^{[n-2]} \circ \exp_b^{[n-2]}
\end{aligned}$$

8.4 Numerics

Repeated exponentiation quickly exhaust the range of floating point numbers. For example for $b = e$, $x = 1/8$ there are at most 4 iterations presentable in extended floating point: $\exp^{[4]}(1/8) \approx 4.9 \times 10^9$. For other numbers like $x = 1/16$ we can obtain 5 iterations in extended floating point arithmetics: $\exp^{[5]}(1/16) \approx 7.7 \times 10^{33555749}$.

So we try to avoid taking too much exponentiations. We do this by not using the formula $\mu_{a,b,n} = \log_a^{[n]} \circ \exp_b^{[n]}$ but save two iterations by using TODO[ref above]:

$$\mu_{a,b,n}(x) = \tau(\mathrm{logi}_{b,a}^{[n-2]}(\exp_b^{[n-2]}(x))).$$

The precision of $\mu_{a,b}(x)$ is limited to the maximal number n of iterations of $\exp_b$ that is presentable in the floating point arithmetic. For example at most 4 iterations are possible in the case $x = 1/8$, which reaches an accuracy of at most $\mu_{\eta,e,6}(1/8) - \exp_\eta(\mu_{\eta,e,6}(\log(1/8))) = \mu_{\eta,e,6}(1/8) - \mu_{\eta,e,5}(1/8) \approx 8.65 \times 10^{-13}$ for $a = \eta$ which is not even "double" precision. For $x = 1/16$ however 5 iterations are presentable in extended floating point arithmetics which gives a nearly exact result.

8.5 Incompatibilities with the other methods

8.5.1 Incompatibility with regular iteration

Looking at equation (8.3) one might ask whether slog_b is the regular superlogarithm (up to an additive constant) if slog_a is. For the appropriate bases $b \in (1, \eta]$ case 1b applies and so slog_a must be defined on (a^+, ∞). This is not the case for the regular superlogarithm at the *lower* fixed point, however it is the case for the regular superlogarithm $\mathrm{slog}_a^{\mathfrak{R}^+}$ at the *upper* fixed point a^+! So we concretize our question to: Is

$$\mathrm{slog}_b^{\mathfrak{R}^+}(x) - \mu_{a,b}(\mathrm{slog}_a^{\mathfrak{R}^+}(x))$$

a constant function for all $a, b \in (1, \eta]$? And the answer is "no", as the graph of this quantity shows in figure 3. The deviation from a constant is small, however the picture show the typical extending oscillation one would expect for different Abel functions. If we have two Abel functions α_1 and α_2 then $\theta = \alpha_1 \circ \alpha_2^{[-1]} - \mathrm{id}$

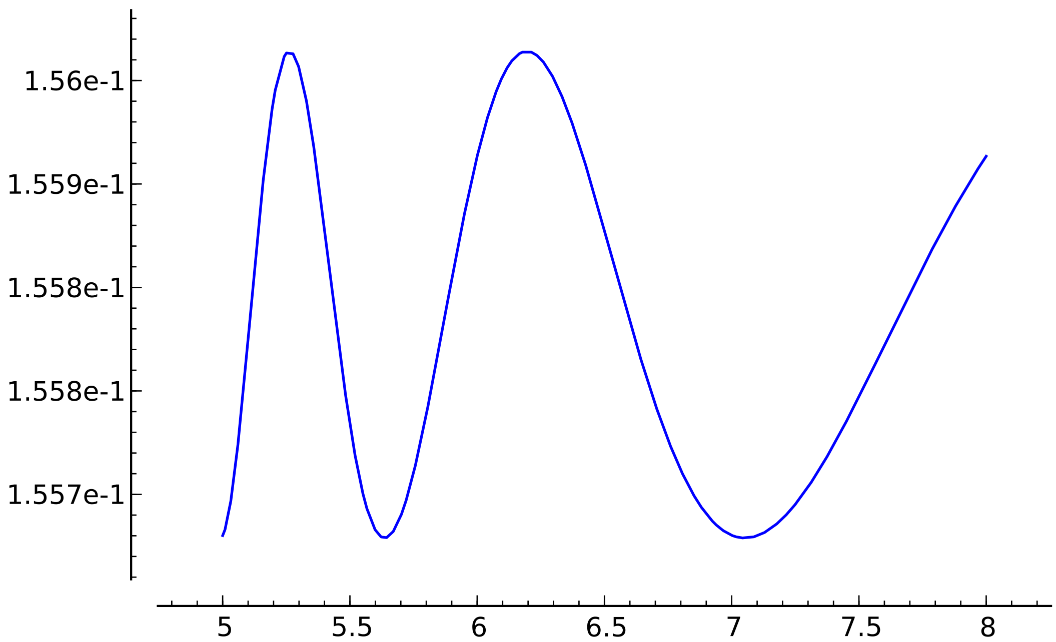

Figure 3: $y = \mathrm{slog}_b^{\mathfrak{R}^+}(x) - \mu_{a,b}(\mathrm{slog}_a^{\mathfrak{R}^+}(x))$ for $a = 8^{1/8}$, $b = 4^{1/4} = \sqrt{2}$

is an 1-periodic function and we can express the difference as $\alpha_2 - \alpha_1 = (\theta + \mathrm{id}) \circ \alpha_1 - \alpha_1 = \theta \circ \alpha_1$. This is an oscillating function with "extending period". One full oscillation takes place in each interval $(x, f(x))$, in our case this is the interval (x, b^x). For example in the picture you can see the two maxima at $x \approx 5.27$ and $4^{5.27/4} \approx 6.2$ or the two minima at $x \approx 5.65$ and $4^{5.65/4} \approx 7.1$.

8.5.2 Incompatibility with intuitive iteration

TODO $\mathrm{slog}_\eta^{\mathfrak{R}^+} \circ \mu_{\eta,e} \neq \mathrm{slog}^{\mathfrak{I}}$

8.5.3 Incompatibility with Cauchy iteration

TODO $\mu_{e,\eta} \circ \mathrm{sexp}_\eta^{\mathfrak{R}^+} \neq \mathrm{sexp}^{\mathfrak{C}}$

9 Summary

Method	Section	Base b	PS devel.	Outcome
Regular ($0 < \lvert f_1 \rvert \neq 1$, PS)	2.1.1	$< \eta$	at FP	iterate
Regular ($0 < \lvert f_1 \rvert < 1$, limit)	2.2.1	$< \eta$	-	Schröder, also iterate
Regular ($f_1 = 1$, PS)	2.1.2	$= \eta$	at FP	iterate
Regular ($f_1 = 1$, limit)	2.2.2	$= \eta$	-	Abel
Matrix Power	3	-	everywhere	iterate
Newton/Lagrange	4	$\leq \eta$	-	iterate
Kneser's	5	$> \eta$	-	Abel
Intuitive Abel	6	-	at non-FP	Abel
Cauchy integral	7	$> \eta$	-	super
Change of base	8	$> \eta$	-	super

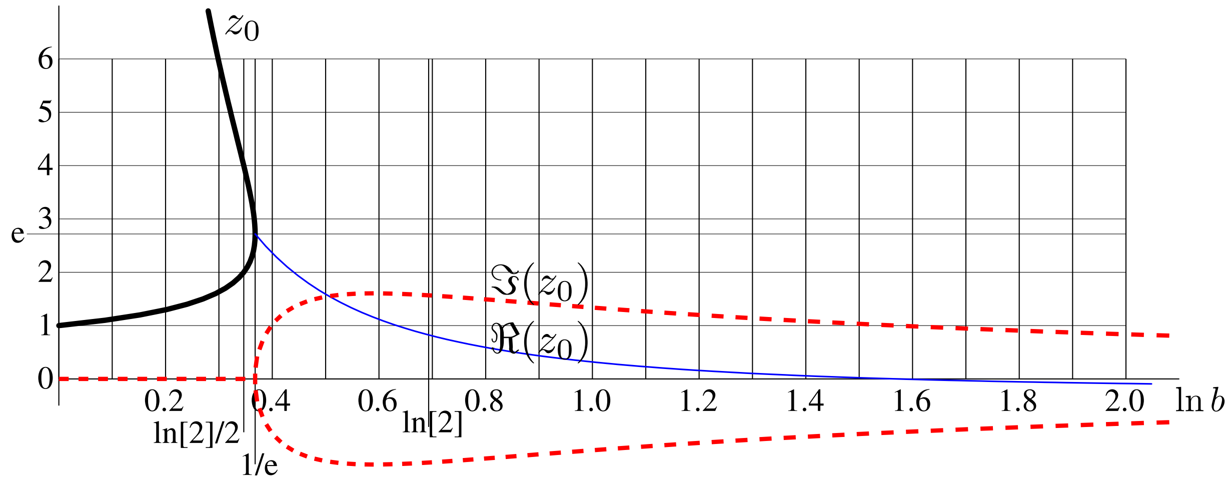

Figure 4: Solutions z_0 of equation $b^{z_0} = z_0$ versus $\ln(b)$, thick solid curve shows two solutions at $\ln(b) < 1/\mathrm{e}$. The thin curve shows the real part of the solution at $\ln(b) > 1/\mathrm{e}$. Dashed curve shows the imaginary part of two solutions, z_0 and z_0^*. [fig02]

PS = power series, FP = fixed point, bases are always greater 1

A Closed Form of the Real Fixed Points of the Exponentials

Standard Knowledge 14 (real fixed points of exponentials)**.** *The real function $f(x) = b^x$ has exactly 2 fixed points for $1 < b < \eta$, exactly one fixed point for $b = \eta$ and no fixed point for $b > \eta$. $b\,[4]\,n$ converges to the lower real fixed point in the case $1 < b \le \eta$ and to infinity for $b > \eta$.*

A fixed point z_0 satisfies $b^{z_0} = z_0$, or equivalently $b = \sqrt[z_0]{z_0}$. This solution versus b is shown in figure 4. To compute z_0 directly we would use the inverse function of $x^{1/x}$. However in most computer algebra systems this function is not implemented, but instead we have the Lambert W function. We use this occasion to investigate the relation of the functions self root, self power and multiplied exponential:

$$\begin{array}{lll}
\mathrm{sr}(x) = x^{1/x} = \sqrt[x]{x} & \mathrm{sp}(x) = x^x & M(x) = xe^x \\
\mathrm{sr}\colon (0,\infty) \to (0,\eta] & \mathrm{sp}\colon (0,\infty) \to [1/\eta,\infty) & M\colon (-\infty,\infty) \to [-1/e,\infty) \\
\mathrm{sr}(e) = \eta \quad (\max) & \mathrm{sp}(1/e) = 1/\eta \quad (\min) & M(-1) = -1/e \quad (\min)
\end{array}$$

and their inverses. All these functions have exactly one local and at the same time global extremum (see figure 5), so there is always a strictly decreasing and a strictly increasing inverse function, which we want to denote by f_-^{-1} and f_+^{-1} respectively.

$$\begin{array}{lll}
\mathrm{sr}_+^{-1}\colon (0,\eta] \to (0,e] & \mathrm{sp}_+^{-1}\colon [1/\eta,\infty) \to [1/e,\infty) & W_+\colon [-1/e,\infty) \to [-1,\infty) \\
\mathrm{sr}_-^{-1}\colon (1,\eta] \to [e,\infty) & \mathrm{sp}_-^{-1}\colon [1/\eta,1) \to (0,1/e] & W_-\colon [-1/e,0) \to (-\infty,-1]
\end{array}$$

Each function is analytically conjugate to each other in the following way:

$$\begin{array}{lll}
\mathrm{sr}(x) = 1/\,\mathrm{sp}(1/x) & \mathrm{sp}(x) = \exp(M(\ln(x))) & M(x) = \ln(\mathrm{sp}(\exp(x))) \\
\mathrm{sr}(x) = \exp(-M(-\ln(x))) & \mathrm{sp}(x) = 1/\,\mathrm{sr}(1/x) & M(x) = -\ln(\mathrm{sr}(\exp(-x)))
\end{array}$$

Accordingly the inverse functions can be defined:

$$\begin{array}{lll}
\mathrm{sr}_\pm^{-1}(x) = 1/\,\mathrm{sp}_\mp^{-1}(1/x) & \mathrm{sp}_\pm^{-1}(x) = \exp(W_\pm(\ln(x))) & W_\pm(x) = \ln(\mathrm{sp}_\pm^{-1}(\exp(x))) \\
\mathrm{sr}_\pm^{-1}(x) = \exp(-W_\pm(-\ln(x))) & \mathrm{sp}_\pm^{-1}(x) = 1/\,\mathrm{sr}_\mp^{-1}(1/x) & W_\pm(x) = -\ln(\mathrm{sr}_\pm^{-1}(\exp(-x))).
\end{array}$$

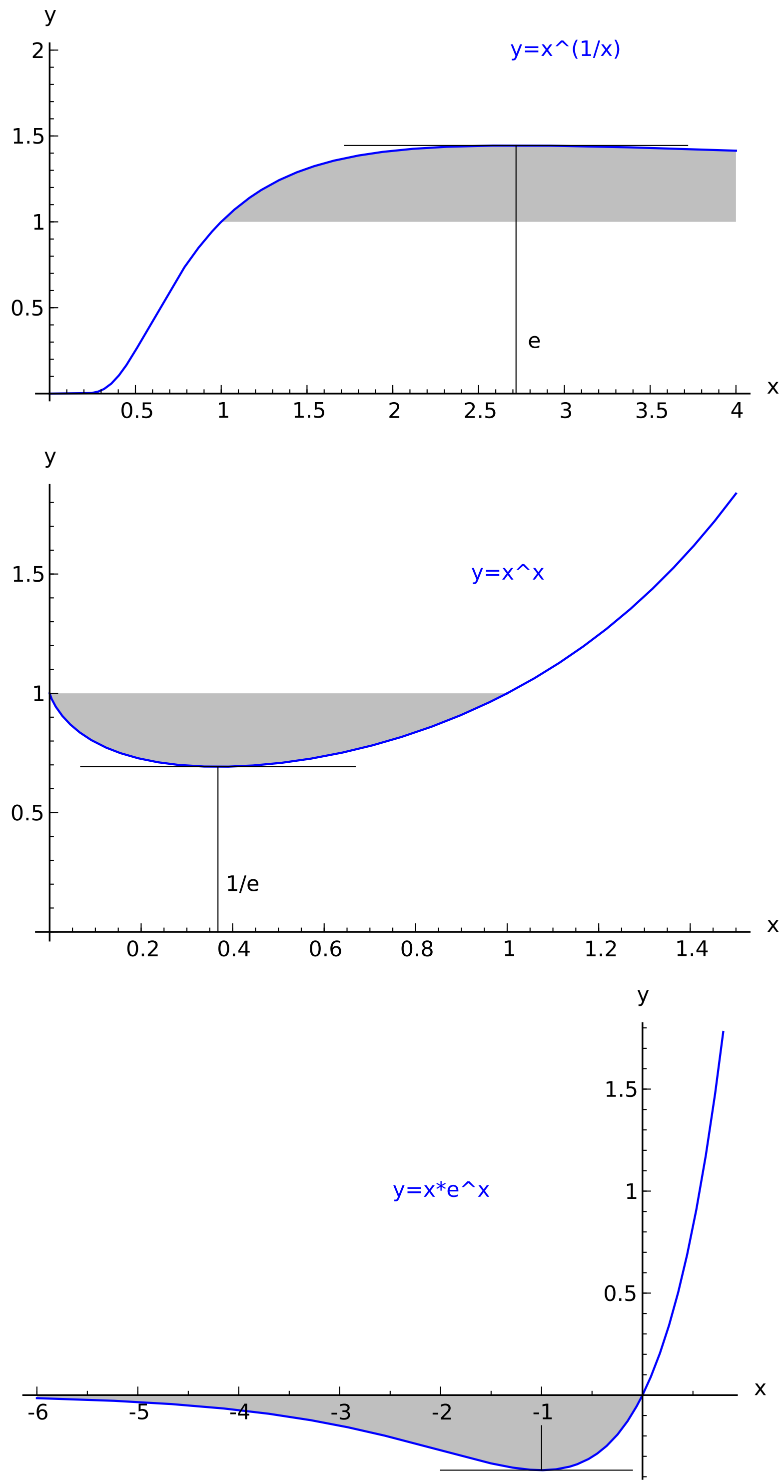

Figure 5: The functions $y = \sqrt[x]{x}$, $y = x^x$ and $y = xe^x$.

We see that each of self root, self power and multiplied exponential can serve as a base function, whose inverse can be used to define the inverses of the other functions. As far as we know of these only the inverse of the multiplied exponential (the Lambert W function) is implemented in standard computer algebra systems. Its branches are indexed equally in these systems and correspond to $W_\pm$ in the following way:

	Maple™	Mathematica™	Sage™
$W_+(z) =$	LambertW(0,z)	ProductLog[0,z]	mpmath.lambertw(z,0)
$W_-(z) =$	LambertW(-1,z)	ProductLog[-1,z]	mpmath.lambertw(z,-1)

Standard Knowledge 15. *The lower fixed point $b^- \in (0,e]$ for $b \in (0,\eta]$ and the upper fixed point $b^+ \in [e,\infty)$ for $b \in (1,\eta]$ of b^x are respectively given by*

$$b^\pm = \mathrm{sr}_\mp^{-1}(b) = 1/\,\mathrm{sp}_\mp^{-1}(1/b) = \exp(-W_\mp(-\ln(b))) = \frac{W_\mp(-\ln(b))}{-\ln(b)}. \tag{A.1}$$

The derivative at the fixed points is

$$\exp_b'(b^\pm) = \ln(b^\pm) = -\ln(\mathrm{sp}_\mp^{-1}(1/b)) = -W_\mp(-\ln(b)). \tag{A.2}$$

Proof. The last part of the first equation is due to $e^{-W(y)} = \frac{W(y)}{y}$. The first part of the second equation: $\exp_b'(b^\pm) = \ln(b)\exp_b(b^\pm) = \ln(b)b^\pm = \ln(b^\pm)$. □

The most famous fixed point pair is probably $(2,4)$ for $b = \sqrt{2}$.

B The complex fixed points of exponentials

Proposition 12. *Let $b > 1$ then for each integer $k \geq 2$ there is exactly one solution z of $z = b^z$ in the horizontal strip $2\pi(k-1)/\ln(b) \leq \Im(z) < 2\pi k/\ln(b)$. We call this solution $b[k]$. More specifically it is situated in $2(k-1)\pi/\ln(b) < \Im(z) < (2\pi k - \pi)/\ln(b)$. For $k = 1$ we distinguish 3 cases :*

1. *If $b > e^{1/e}$ then the above is also valid for $k = 1$.*
2. *If $b = e^{1/e}$ then there is exactly one solution for $k = 1$, this solution is $e =: b[1]$;*
3. *If $1 < b < e^{1/e}$ then there are exactly two solutions $b^- < e < b^+ =: b[1]$.*

For each solution $b[j]$, $j \geq 1$, the conjugate is also a solution of the equation which we denote by $b[-j]$. There are no other solutions than the before mentioned.

Proof. Let $z = re^{i\alpha} = r(\cos(\alpha) + i\sin(\alpha))$ and let $c = \ln(b)$ then the fixed point equation is equivalent to the equation system:

$$r = e^{cr\cos(\alpha)} \tag{B.1}$$

$$\alpha = cr\sin(\alpha) \tag{B.2}$$

Let us now substitute $s = cr$ and assume $\alpha \neq 2\pi m$, for any integer $m \geq 0$:

$$\ln(s) - \ln(c) = \ln(r) = s\cos(\alpha)$$

$$s = \frac{\alpha}{\sin(\alpha)}$$

and substituting the second into the first

$$\ln\frac{\alpha}{\sin(\alpha)} - \ln(c) = \alpha\frac{\cos(\alpha)}{\sin(\alpha)}$$

$$f(\alpha) := \ln\frac{\alpha}{\sin(\alpha)} - \alpha\cot(\alpha) = \ln(c)$$

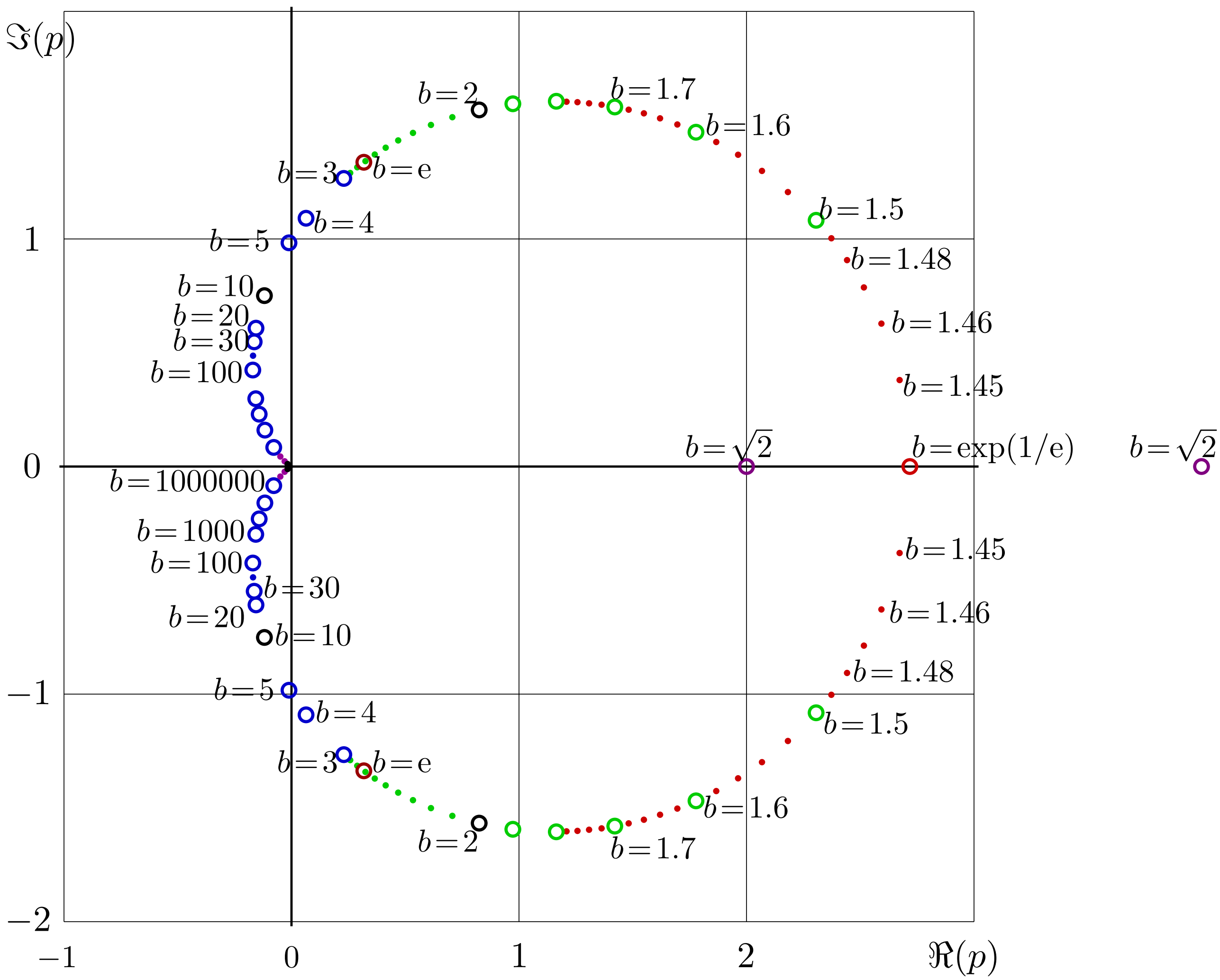

Figure 6: The primary (nearest to the real axis) fixed points p of b^x in the complex p-plane for various values of b. [fig03]

We show that f is strictly increasing on $2k\pi < \alpha < (2k+1)\pi$ for $k \geq 0$ and strictly decreasing for $k < 0$ by contemplating the sign of its derivative:

$$\begin{aligned}
f'(x) &= \frac{\sin(x)}{x}\left(-\frac{x\cos(x)}{\sin(x)^2}+\frac{1}{\sin(x)}\right)-\frac{\cos(x)}{\sin(x)}-x\left(-\sin(x)\frac{1}{\sin(x)}+\cos(x)\frac{-1}{\sin(x)^2}\cos(x)\right)\\
&= -2\cot(x)+\frac{1}{x}+x+x\cot(x)^2 = x+\frac{-2x\cot(x)+1+x^2\cot(x)^2}{x}\\
&= x+\frac{(1-x\cot(x))^2}{x}
\end{aligned}$$

The last line is positive for $x > 0$ and negative for $x < 0$, so there can be at most one solution of $f(x) = \ln(c)$ on $x \in (2k\pi, 2k\pi + \pi)$ and there is a solution for $k \neq 0, -1$ because in this case $f((2k\pi, (2k+1)\pi)) = (-\infty, +\infty)$. The imaginary part of z for this solution is $r\sin(\alpha)$ which is equal to $\alpha/\ln(b)$ by (B.2), so there is exactly one solution in $2k\pi/\ln(b) < \Im(z) < (2\pi(k+1)-\pi)/\ln(b)$ for $k \neq 0, -1$.

Let us now consider the case $k = 0$. Here $f((0,\pi)) = (-1,\infty)$, so if $c > 1/e$ then $\ln(c) > -1$ and there is exactly one solution $f(x) = \ln(c)$ for $0 < x < \pi$. If $\ln(c) \leq -1$ then $f(x) = \ln(c)$ has no solution in $0 < x < \pi$.

We now consider the case $\alpha = 2k\pi$, $k \geq 0$. If $k > 0$ then (B.2) is invalid. So we consider $\alpha = 0$ for which equation (B.2) is always valid. In this case only $r = e^{cr}$ has to be satisfied. Now we look for zeros on $0 \leq x < \infty$ of the corresponding $g(x) = e^{cx} - x$. We can determine the global minimum μ at x of this function by

$$\begin{aligned}
0 &= g'(x) = ce^{cx}-1\\
x &= \frac{1}{c}\ln\frac{1}{c} = -\frac{\ln(c)}{c}\\
g''(x) &= c^2e^{cx} = c^2\frac{1}{c} = c > 0\\
\mu := g(x) &= \frac{1}{c}(1+\ln(c))
\end{aligned}$$

Clearly the minimum μ is smaller than 0 for $\ln(c) < -1$ and equal to 0 for $\ln(c) = -1$ which corresponds to the last both cases $c = \frac{1}{e}$ having one solution and $0 < c < \frac{1}{e}$ having two solutions. $\square$

Proposition 13 (Repelling and attracting fixed points of b^z)**.** *Let $b > 1$, then $|\exp_b{}'(p)| > 1$ for all non-real fixed points p. For the real fixed points in the case $1 < b < e^{1/e}$ we have*

$$\exp_b{}'(b^-) < 1 \qquad\qquad \exp_b{}'(b^+) > 1. \tag{B.3}$$

and $\exp_b{}'(e) = 1$ for $b = e^{1/e}$.

Proposition 14. *Let $b > e^{1/e}$ and let*

$$\log_{b,k}(z) := \frac{\log(z)+2\pi ik}{\ln(b)}$$

then for any $k > 0$ and z_0 in the upper halfplane:

$$\begin{aligned}
b[k] &= \lim_{n\to\infty}\log_{b,k-1}{}^{[n]}(z_0)\\
\exp_b{}'(b[k]) &= \log_{b,k-1}(b[k]).
\end{aligned}$$

The fixed points can in the same way obtained as in knowledge 15 via the branches of sr, sp and Lambert W, where the last is most important practically, as Lambert W is implemented (with its branches) in most computer algebra systems

	Maple$^{\text{TM}}$	Mathematica$^{\text{TM}}$	Sage$^{\text{TM}}$
$W_k(z) =$	LambertW(k,z)	ProductLog[k,z]	mpmath.lambertw(z,k)

With this branching we obtain for $k \geq 1$

$$b[k] = \frac{W_{-k}(-\ln(b))}{-\ln(b)} \qquad 1 < b \tag{B.4}$$

$$b^- = \frac{W_0(-\ln(b))}{-\ln(b)} \qquad b^+ = \frac{W_{-1}(-\ln(b))}{-\ln(b)} \qquad 1 < b \leq \eta. \tag{B.5}$$

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TODO: keywords: exp b fixed points, Kneser Abel function Matrix power 3-argument Ackermann function regular iteration Schroeder function fractional iteration formal powerseries