Uniqueness of Analytic Abel Functions in Absence of a Real Fixed Point

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Abstract. We give a simple uniqueness criterion (and some derived criteria) for holomorphic Abel functions and show that Kneser's real analytic Abel function of the exponential is subject to the criterion.

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1. Introduction

There is a lot of discussion about the "true" fractional iterates of the function e^x in the mathematical community. In 1949 Kneser [3] proved the existence of real analytic fractional iterates. However Szekeres (a pioneer in developing the theory of fractional iteration [8]) states 1961 in [9]:

"The solution of Kneser does not really solve the problem of 'best' fractional iterates of e^x*. Quite apart from practical difficulties involved in the calculation of Kneser's function on the real axis, there is no indication whatsoever that the function will grow more regularly to infinity than any other solution. There is certainly no uniqueness attached to the solution; in fact if* $g(x)$ *is a real analytic function with period 1 and* $g'(x)+1>0$ *(e.g.* $g(x)=\frac{1}{4\pi}\sin(2\pi x)$ *then* $B^*(x)=B(x)+g\big(B(x)\big)$ *is also an analytic Abel function of* e^x *which in general yields a different solution of the equation."*

A recent discussion with Prof. Jean Écalle supports the impression that no uniqueness criterion was found up today and that there is even evidence against the existence of a criterion concerned with the growth-scale or asymptotic behavior at infinity.

By withdrawing our attention from the purely real analytic behavior of the Abel function to the behavior in the complex plane we can succeed in giving a simple uniqueness criterion for Abel functions of a whole class of real analytic (or arbitrary holomorphic) functions with two complex fixed points.

We show the usefulness of the criterion by providing an Abel function that satisfies the criterion. This is the above mentioned by Kneser constructed Abel function of e^x (which can be easily generalized to functions b^x with $b > \mathrm{e}^{1/\mathrm{e}}$).

We have also a suggestion to numerically compute this Abel function and the corresponding fractional iterates of e^x (also of b^x for $b > \mathrm{e}^{1/\mathrm{e}}$ in generalization) by a method developed in [4]. Several other methods to numerically compute holomorphic fractional iterates of e^x or the holomorphic Abel function have emerged in the past years (for example one is given in [11]). A future research goal would be to put them on a thorough theoretic base (proving convergence and holomorphy) and to verify the here given uniqueness criterion.

2. Motivation

Our original motivation was the investigation of a fourth stage of operations after the third stage containing power, exponential and logarithm.

Different terms for such operations were used in the past like: "generalized exponential" and "generalized logarithm" by Walker [10], "ultra exponential" and "infra logarithm" by Hooshmand [2], "super-exponential" by Bromer [1], tetration and superlogarithm [4]. In this paper we give them the more succinct names "4-exponential" and "4-logarithm".

Definition 1 (4-exponential). A *4-exponential* to base $b > 0$ is a function f that satisfies

$$f(0) = 1 \tag{1}$$

$$f(z+1) = \exp_b\big(f(z)\big) \tag{2}$$

for all applicable z.

For any $\tilde{f}$ that only satisfies (2) and contains 1 in its codomain: $\tilde{f}(z_0) = 1$, the function $f(z) = \tilde{f}(z+z_0)$ is a 4-exponential.

Definition 2 (4-logarithm). A *4-logarithm* to base $b > 0$ is a function g that satisfies the Abel equation (4) (see [6]) (for all applicable z) with the following initial condition:

$$g(1) = 0 \tag{3}$$

$$g(\exp_b(z)) = g(z) + 1. \tag{4}$$

For any $\tilde{g}$ that only satisfies (4) and has 1 in its domain of definition, the function $g(z) = \tilde{g}(z) - \tilde{g}(1)$ is a 4-logarithm. Here we set as usual $\exp_b(z) = b^z = \exp\big(\ln(b)z\big)$. The inverse of a 4-exponential (if existing) is a 4-logarithm and vice versa.

On positive integer arguments $z=n$ any 4-exponential f is already determined to be just the n-times application of $\exp_b$ to 1.

$$f(n)=\exp_b^{\circ n}(1)=\underbrace{b^{\cdot^{\cdot^{\cdot^{b}}}}}_{n\times b} \tag{5}$$

The question however is how to properly extend the function real and analytic to non-integer arguments.

The existence of a real analytic strictly increasing 4-logarithm was proven by Kneser [3]. A non-analytic solution with a uniqueness criterion was given by Hooshmand in [2]. A numerical method to compute the real coefficients of the powerseries development at 0 of a 4-logarithm was given (though without proof of convergence) by Walker in [11]. Another numerical method to compute a real analytic 4-exponential via Cauchy integrals (though also without convergence proof) was given by Kouznetsov in [4].

A real analytic 4-exponential is expected to have a singularity or branchpoint at integers ≤ -2 at least on some branch, because from $f(z+1)=\exp_b(f(z))$ follows $f(z-1)=\log_b(f(z))$ and by $f(0)=1$ is then $f(-1)=0$ and $f(-2)=\log_b(0)$. To exclude branching we restrict 4-exponentials to

$$C_{-2}=\mathbb{C}\backslash\{x\in\mathbb{R}: x\le -2\} \tag{6}$$

It is a conjecture of the authors that holomorphy on the domain C_{-2} together with $f(z^*)=f(z)^*$ (complex conjugation) on C_{-2} implies the uniqueness of the 4-exponential f.

From considerations about the uniqueness of 4-logarithms/4-exponentials the following general uniqueness criterion for Abel functions with two complex fixed points emerged.

3. The Uniqueness Criterion

Before we start we mention some *conventions* we use: Usually curves here are regarded as continuous maps on the open interval $(-1,1)$. If we however use a curve in a set context then we refer to the image of the curve, e.g. $\gamma_1\cup\gamma_2=\gamma_1((-1,1))\cup\gamma_2((-1,1))$. The disjoint union $C=A\uplus B$ means here that $C=A\cup B$ and $A\cap B=\emptyset$. The sum $A+z$ of a region $A\subseteq\mathbb{C}$ and a number $z\in\mathbb{C}$ is defined as the region $\{a+z\colon a\in A\}$. A function being holomorphic on a non-open set means that there is a neighborhood of each point of the set where the function is holomorphic. $\log_b(z)=\log(z)/\log(b)$ means the principal branch $-\pi<\Im(\log(z))\le\pi$ of the logarithm if not stated otherwise. “Continuable”, “continuation” and “continue” always refer to *analytic* continuation.

Definition 3 (Abel function, initial curve/region). We call a function α holomorphic on D an *Abel function of F* iff it satisfies the Abel equation

$$\alpha(F(z))=\alpha(z)+1 \tag{7}$$

for all $z\in D\cap F(D)$. F is sometimes called the *base function.*

A curve $\gamma\colon(-1,1)\to\mathbb{C}$ on which F is holomorphic is called an *initial curve of* F iff γ and $F\circ\gamma$ are injective and disjoint and $\gamma(-1)\neq\gamma(1)$ are two fixed points of F. (To be strict define $\gamma(\pm1):=\lim_{t\to\pm1}\gamma(t)$.)

Under these conditions $\gamma\cup(F\circ\gamma)\cup\{\gamma(-1),\gamma(1)\}$ is a closed Jordan curve. We call its inner (bounded) component C joined with γ and $F\circ\gamma$ the *initial region* of γ denoted by $I_F(\gamma):=\gamma\uplus C\uplus F(\gamma)=\overline{C}\setminus\{\gamma(-1),\gamma(1)\}$.

Theorem 1. *Let F be a holomorphic function, let γ be an initial curve of F, let H be its initial region and $d\in H$. There is at most one function α satisfying Criterion 1.*

Criterion 1. The function α is an on H holomorphic and injective Abel function of F, $\alpha(d)=0$ and $\bigcup_{k\in\mathbb{Z}}(\alpha(H)+k)=\mathbb{C}$.

Proof. Assume there are two such Abel functions $\alpha_1\colon H\leftrightarrow T_1$ and $\alpha_2\colon H\leftrightarrow T_2$ holomorphic and injective on H. For the rest of this proof we write α_j when referring to α_1 as well as to α_2. The inverse function $\alpha_j^{-1}\colon T_j\leftrightarrow H$ satisfies:

$$\alpha_j^{-1}(z+1)=F\big(\alpha_j^{-1}(z)\big)$$

for all z such that $z,z+1\in T_j$. So we have two biholomorphic functions

$$q_1:=\alpha_2\circ\alpha_1^{-1}\colon T_1\leftrightarrow T_2 \qquad\qquad q_2:=\alpha_1\circ\alpha_2^{-1}\colon T_2\leftrightarrow T_1$$

with the property

$$\begin{aligned}q_1(z+1)&=\alpha_2\big(\alpha_1^{-1}(z+1)\big)=\alpha_2\big(F\big(\alpha_1^{-1}(z)\big)\big)\\&=\alpha_2\big(\alpha_1^{-1}(z)\big)+1=q_1(z)+1\end{aligned}$$

for each z with $z,z+1\in T_1$; and generally

$$q_j(z+1)=q_j(z)+1 \tag{8}$$

for each z with $z,z+1\in T_j$.

We define $q_{j,k}\colon T_j+k\to\mathbb{C}$ by $q_{j,k}(z+k)=q_j(z)+k$. By our property $q_j(z+1)=q_j(z)+1$ the function $q_{j,k}$ and $q_{j,k+1}$ coincide on the intersection $(T_j+k)\cap(T_j+k+1)$ which contains the curve $\alpha\circ\gamma+k+1$. In conclusion q_j can be continued to the whole complex plane. So lets consider q_j to be an entire function.

For $z\in T_1$ we have $q_1(z)=q_2^{-1}(z)$. But the only entire functions that have an entire inverse are linear functions. By the values $q_1(0)=0$ and $q_1(1)=1$ it can only be the identity. So $\alpha_2(\alpha_1^{-1}(z))=z$ for $z\in T_1$ and hence $\alpha_2=\alpha_1$ on H. $\square$

Now one may argue that the Abel function may depend on the initial region. This is not the case as shown by the next theorem.

Theorem 2. *Let D be a connected set with $d\in D$ and let F be holomorphic on D, there can be at most one on D holomorphic Abel function α of F with the property that $\alpha'(d)\neq0$ and that α satisfies Criterion 1 on some initial region $H\subset D$.*

Proof. Assume that there were two such Abel functions α_1 and α_2 such that each α_j is holomorphic on D and satisfies Criterion 1 on the initial region H_j. We follow the proof of Theorem 1, listing only the modifications.

We consider biholomorphic $\alpha_j\colon H_j \leftrightarrow T_j$ and holomorphic $q_j\colon T_j \to \mathbb{C}$ and achieve (8). In conclusion each q_j is entire. The function $z \mapsto q_1(\alpha_1(z)) - \alpha_2(z)$ is holomorphic on D. It is constantly 0 on H_1 and hence on D. That's why $q_1(0) = q_1(\alpha_1(d)) = \alpha_2(d) = 0$. Vice versa $q_2(0) = 0$.

There is a neighborhood V of d where α_1 and α_2 is injective. Hence α_j^{-1}, $q_1 = \alpha_2 \circ \alpha_1^{-1}$ and $q_2 = \alpha_1 \circ \alpha_2^{-1}$ are injective on $V' = \alpha_1(V) \cap \alpha_2(V) \ni 0$. But then $q_1 = q_2^{-1}$ on $V' \cap q_2(V')$. $\square$

Now we want to make the criterion $\bigcup_{k\in\mathbb{Z}}(\alpha(H)+k) = \mathbb{C}$ a bit more accessible and show that it is a consequence of $\lim_{t\to\pm 1} \Im(\alpha(\gamma(t))) = \pm\infty$. Before we start with the actual proof, we need a little insight into how curves divide the complex plane. We know by the Jordan curve theorem that each simple closed curve divides the sphere into two simply connected components. Considering the sphere $\mathbb{C} \cup \{\infty\}$ we know that each injective curve $\zeta\colon (-1,1) \to \mathbb{C}$ with $\lim_{t\to\pm 1} \zeta(t) = \infty$ divides the complex plane into two parts; where ∞ is the complex infinity and $\lim_{t\to\pm 1}\zeta(t) = \infty$ means that for each $r > 0$ there is a t_1 and t_0 such that $|\zeta(t)| > r$ for all $t > t_1$ and all $t < t_0$.

A particular subclass of such curves are the injective curves ζ with $\lim_{t\to\pm 1} \Im(\zeta(t)) = \pm\infty$. Here ∞ is the real infinity and $\lim_{t\to\pm 1}\Im(\zeta(t)) = \pm\infty$ means that for each $u \in \mathbb{R}$ there are $t_0, t_1 \in (-1,1)$ such that $\Im(\zeta(t)) > u$ for all $t > t_1$ and $\Im(\zeta(t)) < u$ for all $t < t_0$.

Definition 4 (left/right component/ray). For an injective curve $\zeta\colon (-1,1) \to \mathbb{C}$ we call a component *left* (resp. *right*) if it contains a left (resp. right) ray, where a left (resp. right) ray is a set of the form $\{z_0 \mp x\colon x > 0\}$ for some $z_0 \in \mathbb{C}$.

The following lemma shows that that these properties indeed behave as expected.

Lemma 3 (L, R). *Let $\zeta\colon (-1,1) \to \mathbb{C}$ be an injective curve with $\lim_{t\to\pm 1}\Im(\zeta(t)) = \pm\infty$ and let P and Q the two components the plane is divided into; $\mathbb{C} = P \uplus \zeta \uplus Q$.*

1) Then either P is left and Q is right or vice versa. We denote the left (resp. right) component with $L(\zeta)$ (resp. $R(\zeta)$). Moreover there exist left (resp. right) rays contained in the left (resp. right) component for each prescribed imaginary part y.

2) If $\zeta + d$ is disjoint from ζ then $\overline{R}(\zeta + d) \subset R(\zeta)$ in the case $d > 0$ and $\overline{L}(\zeta + d) \subset L(\zeta)$ in the case $d < 0$. Here the overline means the closure of the component (it is equal to the union of the component with ζ).

Proof. For part 1) of the lemma we show that for each prescribed imaginary part y there exists a left ray contained in one component and a right ray contained in the other component. We show further that the union of all left (resp. right) rays such that each is contained in either P or Q must be contained in either P or Q, which ensures the "either" in part 1).

We consider the indices of the intersection of the horizontal line $Y = \{x+iy\colon x\in\mathbb{R}\}$ with the curve ζ, $T=\{t\colon \zeta(t)\in Y\}$. Now let $t_0=\inf(T)$ and $t_1=\sup(T)$; neither can $t_0=-1$ nor $t_1=1$ because in this case there would be a sequence of $t\to\pm1$ such that $\Im(\zeta(t))=y$ in contradiction to $\Im(\zeta(t))\to\pm\infty$.

Hence $-1<t_0\le t_1<1$ and $\Im(\zeta(t))>y$ for all $t>t_1$ and $\Im(\zeta(t))<y$ for all $t<t_0$. Let $x_0=\Re(\zeta(t_0))$ and $x_1=\Re(\zeta(t_1))$ then it is clear that $Y_0:=\{x+iy\colon x<x_0\}$ is completely contained in a component as well as $Y_1:=\{x+iy\colon x>x_1\}$ is completely contained in a component. The compound $Y_0\cup\zeta([x_0,x_1])\cup Y_1$ divides the plane into an upper half H_1 which is divided by $\zeta((x_1,1))$ and a lower part H_0 which is divided by $\zeta((-1,x_0))$.

If there was a path $\beta\colon[0,1]\to\mathbb{C}$ that connects Y_0 with Y_1, i.e. $\beta(0)\in Y_0$ and $\beta(1)\in Y_1$ then we choose $s_0=\beta^{-1}(\sup(\Re(\beta\cap Y_0))+\mathrm{i}y)$ and $s_1=\beta^{-1}(\inf(\Re(\beta\cap Y_1))+\mathrm{i}y)$. The restriction of β to the non-empty interval (s_0,s_1) still connects Y_0 to Y_1 but does neither intersect Y_0 nor Y_1. As the path is also not allowed to intersect $\zeta([t_0,t_1])$ it must either be contained in H_0 or in H_1 but then it would intersect $\zeta((-1,x_0))$ or $\zeta((x_1,1))$.

If we have two left rays with the imaginary parts y_1 and y_2 then we can do the above construction of $t_0(y)$ for $y=y_1,y_2$. Without restriction let $t_0(y_1)<t_0(y_2)$, then $\zeta([t_0(y_1),t_0(y_2)])$ has a minimum of the real part and we can connect both left rays by a vertical line with a smaller real part.

Now to part 2: We consider $d>0$. First it is easy to see that one point of $\zeta+d$ lies in $R(\zeta)$ and hence the whole $\zeta+d$ is contained in $R(\zeta)$. Vice versa ζ must be contained in $L(\zeta+d)$. So ζ does not intersect $R(\zeta+d)$ and we conclude that $R(\zeta+d)$ must be either contained in $L(\zeta)$ or in $R(\zeta)$. But $R(\zeta)$ has common points with $R(\zeta+d)$ as a right ray Y is contained in $R(\zeta)$ and the ray $Y+d$ is contained in $R(\zeta+d)$. □

Criterion 2. Under the preconditions of Theorem 1, and γ being rectifiable, the following criterion implies Criterion 1:

The curve $\zeta=\alpha\circ\gamma$ is injective, $\zeta\cap(\zeta+1)=\emptyset$ and $\lim_{t\to\pm1}\Im(\zeta(t))=\pm\infty$; where α is an on H holomorphic Abel function of F with $\alpha(d)=0$.

Proof. The first two conditions of the criterion state that α is injective on $\gamma\cup(F\circ\gamma)$ because $\alpha\circ F\circ\gamma=\zeta+1$ by (7). By a theorem about univalent functions (see [7] Theorem 4.8) the injectiveness of α on the rectifiable boundary with finitely many exceptions (e.g. $\gamma(-1)$ and $\gamma(1)$) implies the injectiveness on the enclosed region (inclusive its boundary) which is H.

Now we show that $\bigcup_{k\in\mathbb{Z}}(\alpha(H)+k)=\mathbb{C}$: The image $\alpha(H)$ is bounded by ζ and $\zeta+1$ and it must be simply connected, that's why

$$\alpha(H)=\overline{R}(\zeta)\cap\overline{L}(\zeta+1)$$

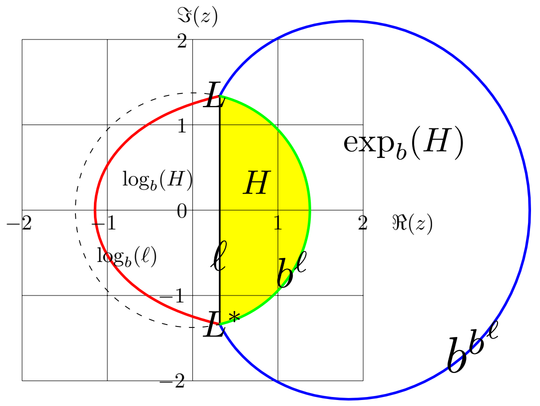

FIGURE 1. Contours $\log_b(\ell)$, ℓ and b^ℓ in the complex z-plane, for base $b = \mathrm{e}$.

If we now unite consecutive pieces $\alpha(H)+k$, $k\in\mathbb{Z}$, we get with $\overline{R}(\zeta-1)\supset \overline{R}(\zeta)$ and $\overline{L}(\zeta)\subset\overline{L}(\zeta+1)$ from Lemma 3 that:

$$\bigcup_{k=k_0}^{k_1}(\alpha(H)+k) = \overline{R}(\zeta+k_0)\cap\overline{L}(\zeta+1+k_1)$$

By part 1) of Lemma 3 we also know that for each $z = x+iy$ there is a (negative) $k_0\in\mathbb{Z}$ such that $z\in R(\zeta+k_0)$ (right ray with starting point $x-k_0+iy$ is contained in $R(\zeta)$) and a (positive) $k_1\in\mathbb{Z}$ such that $z\in L(\zeta+k_1)$ (left ray with starting point $x-k_1+iy$ is contained in $L(\zeta)$). So $\alpha(H)$ translated by all integers cover the whole complex plane:

$$\bigcup_{k\in\mathbb{Z}}(\alpha(H)+k) = \mathbb{C} \tag{9}$$

□

Criterion 3. Under the preconditions of Theorem 1 the following criterion implies Criterion 2:

The real function $f(t) = \Im(\alpha(\gamma(t)))$ is strictly increasing and $\lim_{t\to\pm1} f(t) = \pm\infty$; where α is an on H holomorphic Abel function of F with $\alpha(d) = 0$.

Proof. First it is clear that $\zeta := \alpha\circ\gamma$ is injective as no imaginary value can be taken twice. Further the correlation μ given by $\Im(\zeta(t))\mapsto\Re(\zeta(t))$, $t\in(-1,1)$, is a welldefined function on $\mathbb{R}$. Then $\mu+1>\mu$ is a function not intersecting μ, and hence $\zeta+1$ does not intersect ζ. □

4. Application to Kneser's construction

In this section we apply the uniqueness Criterion 1 to the 4-logarithm (i.e. Abel function of $\exp_b$) Ψ as constructed by Kneser [3].

Related to Kneser's construction we use the choice

$$H = \{z \in \mathbb{C} \colon \Re(z) \geq \Re(L), |z| \leq |L|\} \setminus \{L, L^*\} \tag{10}$$

as initial region which is depicted in Figure 1 for $b = \mathrm{e}$ (which is the only base that Kneser considered). Here L is the fixed point of $\log_b$ in the upper half plane. The straight line

$$\ell(t) = \Re(L) + \mathrm{i}\Im(L)t, \quad -1 < t < 1, \tag{11}$$

between L and its complex conjugate L^* is the left boundary of H and b^ℓ is the right boundary of H.

Lemma 4. *If $b > \mathrm{e}^{1/\mathrm{e}}$ then ℓ in* (11) *is an initial curve and hence H is an initial region of* $\exp_b$.

Proof. We show that b^ℓ is injective and does not intersect ℓ. By $b^L = L$ we know that $b^{\Re(L)} = |b^L| = |L|$ and hence

$$b^{\ell(t)} = b^{\Re(L) + \mathrm{i}\Im(L)t} = |L| \mathrm{e}^{\mathrm{i}\Im(L)\ln(b)t}$$

which is an arc with radius $|L|$ centered in 0 (shown with a dashed line) starting at angle $-\Im(L)\ln(b)$ and ending at angle $\Im(L)\ln(b)$. This is true for any non-real conjugated fixed point pair of $\exp_b$. For $b > \mathrm{e}^{1/\mathrm{e}}$ there are no real fixed points of $\exp_b$ and the fixed point pair of $\log_b$, which is the one closest to the real axis, has $\pm\Im(L) \in (-\pi/\ln(b), \pi/\ln(b))$. This assures that b^ℓ does not overlap itself, i.e. that it is injective. $\square$

Let us — without proof — enumerate some counterexamples of initial curves: $\log_b \circ \ell$ is initial for $e^{1/e} < b < e^{\pi/2}$ but ℓ has zero or negative real part for $b \geq e^{\pi/2}$ and is hence no more contained in the default domain of the logarithm. The curve b^ℓ is initial for $b = \mathrm{e}$ but there are bigger bases where b^{b^ℓ} intersects itself. Each of e^ℓ, $\mathrm{e}^{\mathrm{e}^\ell}$, $\mathrm{e}^{\mathrm{e}^{\mathrm{e}^\ell}}$ are injective but $\mathrm{e}^{\mathrm{e}^{\mathrm{e}^{\mathrm{e}^\ell}}}$ is not, so the first two are initial for exp. The straight line connecting any other conjugated fixed point pair is not initial because the image under $\exp_b$ is a circle with radius $|L|$ (winding at least once around 0).

To get familiar with Kneser's construction we recapitulate the main steps he does in [3] with a slight generalization to bases $b > \mathrm{e}^{1/\mathrm{e}}$. He starts with the Kœnigs function χ of $\exp_b$ at the fixed point L and shows that it can be continued to nearly the whole upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \colon \Re(z) \geq 0\} \setminus \{0, 1, b, b^b, b^{b^b}, \ldots\}$. He also shows that it is injective on $\mathfrak{H}$ because its inverse can be continued to an entire function.

Particularly χ is injective on the "half" initial region

$$\mathfrak{H}_0 = \{z \in \mathbb{C} \colon \Re(z) \geq \Re(L), \Im(z) \geq 0, |z| \leq |L|\} \setminus \{L, 1\}.$$

Note that Kneser initially works with $\mathfrak{H}_0$ containing L while he later switches to consider $\mathfrak{H}_0$ without L which is the definition we use above. $\mathfrak{H}_0$ is related to our initial region H via $\mathfrak{H}_0 \cup \mathfrak{H}_0^* = H \setminus \{1\}$.

The Kœnigs function satisfies the Schröder equation $\chi(b^z)=c\chi(z)$ on $z,b^z\in\mathfrak{H}$ where $c=\exp_b'(L)=\log(L)$. Next he sets $\psi(z)=\log(\chi(z))$ for a suitable region of the logarithm which has the cut outside $\chi(\mathfrak{H}_0)$. It satisfies $\psi(b^z)=\psi(z)+\log(c)$ on $z\in\mathfrak{H}_0$. Last he conformally maps the union of $\psi(\mathfrak{H}_0)+k\cdot\log(c)$, $k\in\mathbb{Z}$, via the Riemann mapping theorem to the upper halfplane, say with the conformal map ϱ.

The resulting function $\Psi=\varrho\circ\log\circ\chi$ is at least defined on $\mathfrak{H}_0$ and satisfies $\Psi(b^z)=\Psi(z)+1$ on $\mathfrak{H}_0$. It can even be continued to $z=1$. It is real analytic on $\mathbb{R}\cap\mathfrak{H}_0$ and can hence analytically continued to the conjugate region $\mathfrak{H}_0^*$ (and also to the whole real line).

From his construction the following things are important for us. First: Ψ is injective and holomorphic on $\mathfrak{H}_0\cup\{1\}$ and the image is contained in the upper halfplane. Hence the conjugate continuation to H is also injective and holomorphic. Second: the by integer translated regions $\Psi(\mathfrak{H}_0\cup\{1\})$ cover the whole upper halfplane and hence the by integer translated regions $\Psi(H)$ cover the whole complex plane.

The application of Theorem 1 gives:

Theorem 5. *The by Kneser in* [3] *constructed real analytic 4-logarithm $\alpha=\Psi$ and each generalization to base $b>\mathrm{e}^{1/e}$ satisfies Criterion 1 on H given in* (10) *(where $d=1$ and $F(z)=b^z$).*

5. About Computation of the 4-Logarithm/4-Exponential

As Kneser uses the Riemann mapping theorem in his construction, it is very difficult to approach computationally. Instead we use here a rather fast numerical method given in [4] utilizing the Cauchy integral formula to compute a 4-exponential which we denote with ksexp_b and its inverse with kslog_b. This method is originally described for $b=\mathrm{e}$ but can be extended to arbitrary bases $b>\mathrm{e}^{1/\mathrm{e}}$. Unfortunately it lacks a proof of convergence (but if it converges then kslog_b satisfies the Abel equation and is holomorphic). It is the conjecture of the authors that this method does converge and that it satisfies the uniqueness Criterion 3.

For real values of the argument, this 4-exponential is plotted in Figure 2 for $b=\mathrm{e},2$. The plot of $\mathrm{kslog}_\mathrm{e}$ in the complex plane (depicted in figure 3) hints towards Criterion 3.

5.1. The Fractional Iterates of the Exponential

The combination of ksexp and kslog allows to define fractional/continuum iterative powers of the exponential via (see e.g. [8] or [6]):

$$\exp_b^{\circ c}(z)=\mathrm{ksexp}_b\big(c+\mathrm{kslog}_b(z)\big) \tag{12}$$

Unlike regular iterates at a hyperbolic fixed point which are analytic at the fixed point, the above iterates of $\exp_b$ have a branch point at both complex fixed points. Care must be taken to determine a principal branch/region/cut of kslog_b which is then also the principal region of the iterative power. It

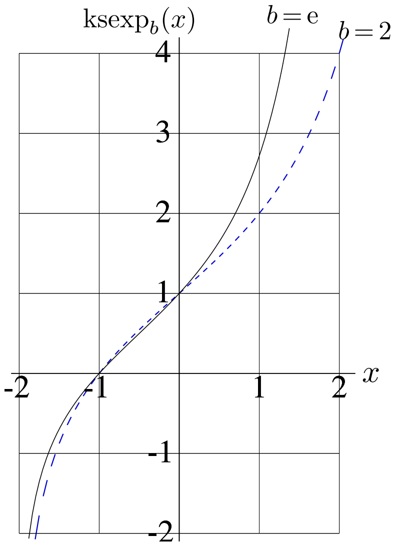

FIGURE 2. 4-exponential ksexp_b to base $b = \mathrm{e}$ (thick solid) and $b = 2$ (dashed) on the real axis.

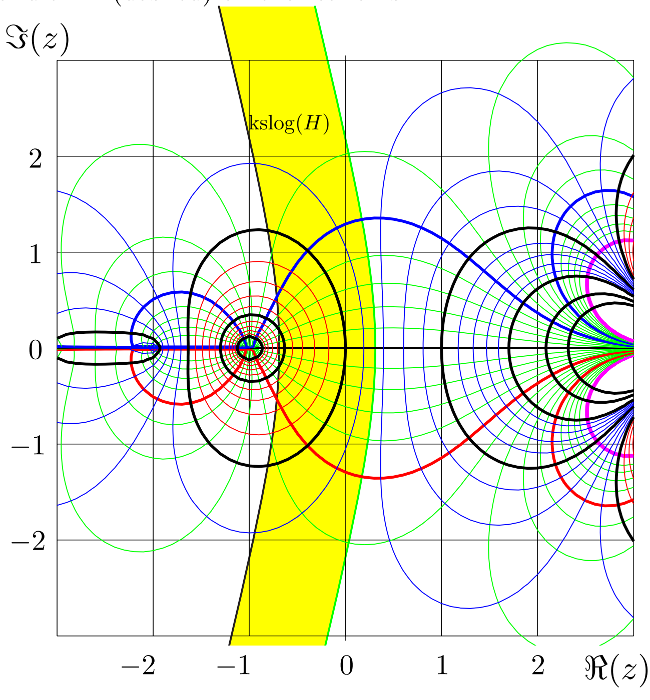

FIGURE 3. Contour plot of the function $\mathrm{ksexp}_{\mathrm{e}}(z)$ showing lines of constant modulus and of constant phase. The shaded (yellow) region is $\mathrm{kslog}_{\mathrm{e}}(H)$. Its right boundary is $\mathrm{kslog}_{\mathrm{e}}\left(\mathrm{e}^{\ell}\right)$ which corresponds to $|\mathrm{ksexp}_{\mathrm{e}}(z)| = |L|$.

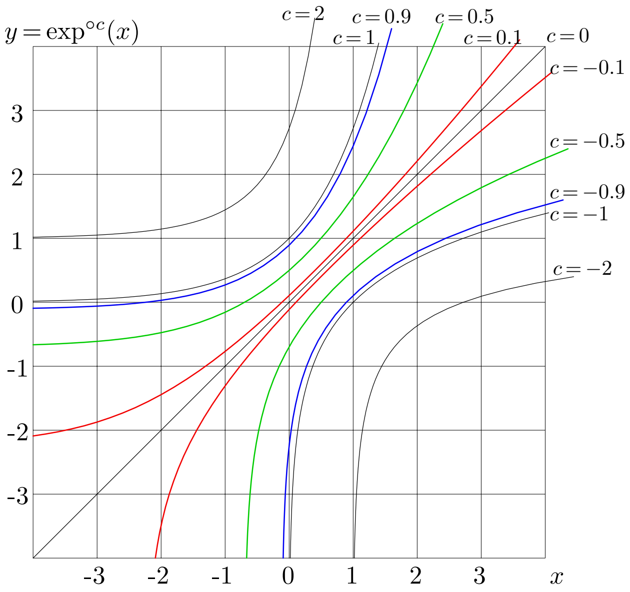

FIGURE 4. Function $y=\exp^{\circ c}(x)$ calculated by equation (12) for $c=0$, ± 0.1, ± 0.5, ± 0.9, ± 1, ± 2 versus x.

should be chosen such that $c+\mathrm{kslog}_b(z) \in C_{-2}$. The map of the function $\exp^{\circ 0.5}$ in the complex plane is plotted in reference [5], showing a behavior similar to the iterative square root of the factorial $\sqrt{!}$ computed in [5].

For several *real* c we plot $\exp^{\circ c}(x)$ versus x in Figure 4. For $c=1$ it is indeed the usual exponential, for $c=0$ it is the identity function, and for $c=-1$ it is the logarithm.

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Notations

symbol	meaning and first occurrence
* (asterisk in superscript)	complex conjugation (except introduction)
$\overline{}$ (overline)	topological closure, Lemma 3
$\cup$	the union of sets, see Definition 3, Criterion 1
$\uplus$	disjoint union; $C=A\uplus B$ means that $C=A\cup B$ and $A\cap B=\emptyset$
$\cap$	the intersection of sets
$\emptyset$	the empty set
b	base of the exponential ($b>1$) e.g. $\exp_b(z)=b^z$ (2)
$\mathbb{C}$	the set of complex numbers
C_{-2}	possible complex domain of definition of 4-exponentials (6)
C	The interior of the initial region: Definition 3
d	a complex constant
$\exp_b(z)=b^z=\exp\big(\ln(b)\ z\big)$	exponential to base b
$\exp_b^{\circ c}(z)$	exponential, iterated c times (12) and (5)
e	the Euler constant
F	the base function, Definition 3
f	a function; e.g. a 4-exponential (1), (2)
g	a function; e.g. a 4-logarithm (3), (4)
γ	curve in the domain of the Abel function, Theorem 1
H	initial region, particularly of $\exp_b$ (10), Figure 1
α	an Abel function (4)
i	imaginary unit
$\Im$	imaginary part of its argument
$I_F(\gamma)$	The initial region of the initial curve γ, Definition 3
j	index $=1$,2 (8)
k	integer number
L	the fixed point of $\log_b$ in the upper halfplane, Figure 1
ℓ	an initial curve of $\exp_b$ from L^* to L; particular case of γ, Figure 1
π	the circular constant
P, Q	component sets in the complex plane Lemma 3
q	the "functional" difference/quotient of two Abel functions (8)
$\mathbb{R}$	the set of real numbers
$\Re$	real part of its argument
T_j	codomain of α_j, Criterion 1
t	parameter of curves, Lemma 3
x,y	usually real and imaginary part of a complex number, Lemma 3
z	a complex variable; sometimes $z=x+\mathrm{i}y$
$\mathbb{Z}$	the set of integer numbers
ζ	a curve in the codomain of the Abel function, Lemma 3
Ψ, ψ, χ, $\mathfrak{H}$, $\mathfrak{H}_0$, ϱ	functions and sets in Kneser's description