

Relativity and the Global Positioning System

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FUNDAMENTALS OF GPS

GPS fundamentally relies on the consistency of the speed of light in any inertial frame. Given the positions \vec{r}_i and times t_i of four GPS satellites a receiver can determine their position \vec{r} and time t using:

$$c^2(t - t_i)^2 = |\vec{r} - \vec{r}_i|^2$$

While the relativistic effects on the system are fairly small, the clock on the satellites only need to be out by 3ns for there to be an error of 1m to the calculation of position.

RELATIVISTIC ERRORS

There are 4 main sources of relativistic errors in the GPS:

- Time dilation due to the motion of the satellites
- Gravitational time dilation
- The Sagnac effect
- Doppler shifts in radio frequencies

TIME DILATION DUE TO SATELLITE MOTION

If we consider a stationary observer on (but not moving with) the earth measuring the velocity of a GPS satellite they find that the velocity of a satellite v_s is approximately 3900m/s. Thus by using the special relativistic time dilation formula $\gamma_s = \left(1 - \frac{v_s^2}{c^2}\right)^{-1/2}$ we find that the fractional change in time compared to a motionless clock is 0.85×10^{-10} . Here we can safely ignore the motion of a clock due to the rate of the earth, as it has a corresponding value of γ which is over 100 times smaller than for the satellites.

DOPPLER EFFECT

Due to the high speed of the orbiting satellites relativistic Doppler effects have to be considered when detecting signals from the satellites. The relativistic frequency shift is given by

$$f = f_0 \frac{\sqrt{1 - v_s^2/c^2}}{1 - v_s \cos \alpha}$$

where v_s is the velocity of the satellite, α is the angle of the light ray from the satellite's motion. For the main $L_1 = 1575.42\text{MHz}$ carrier frequency which is used for most of the sending of GPS data this gives potential offsets of up to $\pm 10\text{KHz}$ which the receivers must account for.

SAGNAC EFFECT

The Sagnac effect is caused by the motion of the earth. As an observer on the surface of the earth receives signals from the satellites in orbit we must correct for if the satellite is ahead of the motion of the earth or behind it. Consider the standard Minkowski line element $-ds^2 = -(cdt)^2 + dr^2 + r^2d\phi^2 + dz^2$ in cylindrical coordinates. Transform this to an equivalent flat space but with the azimuthal coordinate rotating with angular speed ω_e :

$$-ds^2 = -\left(1 - \frac{\omega_e^2 r'^2}{c^2}\right)(cdt')^2 + 2\omega_e r'^2 d\phi' dt' + (d\sigma')^2$$

with $(d\sigma')^2 = (dr')^2 + (r'd\phi')^2 + (dz')^2$. Light travels such that $ds^2 = 0$. If we ignore terms $\frac{\omega_e^2 r'^2}{c^2} \ll 1$ we can solve for dt' and integrate to obtain the time of travel:

$$\int_{\text{path}} dt' = \int_{\text{path}} \frac{d\sigma'}{c} + \frac{2\omega_e}{c^2} \int_{\text{path}} dA'_z$$

where dA_z is the projection of the path taken by an observer on the surface of the earth projected onto a triangular infinitesimal area on a plane through the equator. Thus we observe that the Sagnac effect is dependent on how close you are to the equator: for observers at the pole no such effect would be measured, whereas for observers at the equator this could amount to a discrepancy in measured time from the satellite of up to $\pm 207\text{ns}$.

GRAVITATIONAL TIME DILATION

Gravitational time dilation states that a clock in a higher gravitational potential will run faster than a clock in a lower potential. As such there will be an effect contrary to the special relativistic effect slowing the satellite clocks down, due to the satellites being at a higher potential than the surface of the earth. This effect is given by

$$\frac{\Phi_E - \Phi_s}{c^2}$$

with Φ_E the potential at the earth's surface and Φ_s the potential at the satellite. By taking $\Phi = -\frac{GM}{r}$ (the standard Newtonian potential) we find that this effect corresponds to a fractional change in time of -5.2×10^{-10} . This is far larger than the special relativistic result. Thus we see that the GPS depends not only on special relativistic physics but also on general relativity.

REFERENCES

Relativity, Gravitation and Cosmology - Ta-Pei Cheng
Gravity: An introduction to Einstein's Relativity - James B. Hartle
Relativity in the Global Positioning System - Neil Ashby - <http://www.livingreviews.org/lrr-2003-1>
Global Positioning System: Theory and Applications Vol 1 - Neil Ashby and James Spilker - pages 623 - 697

SAGNAC EFFECT DIAGRAM

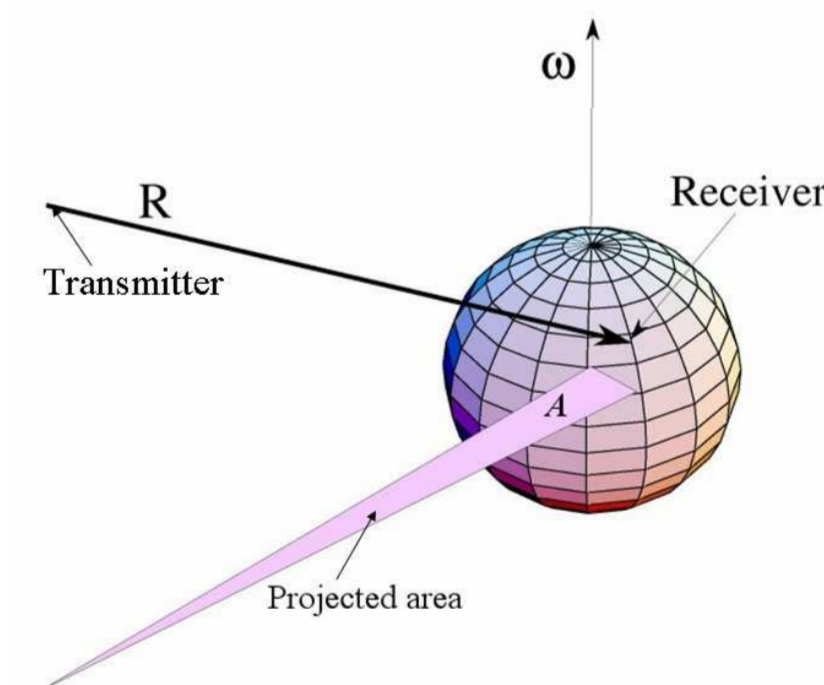


Diagram of the projection onto A_z taken from Ashby's article.

CONCLUSIONS

The Global Positioning System would not be able to function without careful analysis of the physics using relativistic concepts. This poster has only touched on the main issues which relativity brings up: an interested reader is directed to the article by Ashby which describes in depth several other effects which make themselves known in the Global Positioning System.