PROBLEMS IN ANALYSIS AND ALGEBRA

SIDDHARTHA GADGIL

1. Overview

This is a resource for students learning to solve problems that involve finding proofs. The goal here is to present not just problems and proofs, but guidance on how a student should go about attempting to find a proof. We focus not on *tricks* to solve unusual problems but on the basic principles used to find proofs.

Contents

2. A simple analysis problem.

We begin with a problem whose conclusion is *obvious.* Our goal is to see how to find a complete proof.

2.1. **Problem:** Consider the sequence a_n given by:

- $a_n = n/2$ if n is even.
- $a_n = 0$ if n is odd.

Show that this sequence is **not** bounded above.

2.2. **How to solve it:**

2.2.1. *Examining the problem.*

- Firstly, the *hypothesis* - what is given to you, is rather simple. It is just an explicitly described sequence.
- Next, the *conclusion* we need is to show that the given sequence is not bounded.

2.2.2. *Recalling relevant definitions and results.*

- The one definition to recall is that of a bounded sequence. As we want to show that a_n is not bounded above, let us recall what it means for a_n to be bounded above.

" The sequence a_n is bounded above if there exists a real number C so that for all $n \in \mathbb{N}$, we have $a_n < C$. "

- In this case, it turns out that we are best off working straight from the definition. Experience will tell us when to try to use a theorem and which one to use.

2.2.3. *Strategy for proof.* We often have to try different approaches till one succeeds. It is with experience that one learns which approach is most likely to succeed in a given situation, and, crucially, how long to follow up on an approach before changing it.

In this case, we follow a frequently fruitful approach - prove by contradiction. We want to show that the sequence a_n is not bounded, so we assume that it is and work from there.

Assume boundedness: By the definition we just recalled, this means that we assume the following:

"There is a constant $C > 0$ so that for all $n \in \mathbb{N}$, $a_n < C$"

Remember C is fixed, but we do not know what it is. So we look for a contradiction treating C as a fixed but unknown constant. So our assumption is

"for all $n \in \mathbb{N}$, $a_n < C$."

Let us see why this is impossible. Clearly a contradiction is more likely to come from an even n, as for n odd $a_n = 0$, so $a_n < C$ as long as C is positive.

Remark: It is worth observing that here we have used another fruitful strategy, consider cases separately. If we have to find a contradiction, it is enough to find it in one case, so long as that case is not empty. If we had to prove a result, we would of course have to prove it in all cases.

Returning to our goal, we want to show that it is not true that $a_n < C$ for all n. This brings us to one of the most subtle aspects of learning proofs, correctly negating (i.e., finding the opposite meaning of) statements, especially those involving *for all* (i.e., *every*) and *there exists* (i.e., *some*). Let us digress a little to consider such an example:

Negate: *Every man is left-handed.*

Clearly, the negation is *some man is not left-handed.* This is part of a general rule, when negating, *every* becomes *some* and *some* becomes *every.*

Back to our goal: we wanted to show that it is **not** true that for all $n \in \mathbb{N}$, $a_n < C$. In other words, we wanted to show that the negation of this is true. So following the principle of making *every* (*for all*) into *some* (*there exists*), what remains is to show:

"there exists $n > 0$ so that $a_n \geq C$."

As we have already observed, it is better to look for an even n satisfying this. Using the definition of a_n for even n, we end up with

"there exists n even so that $n/2 \geq C$."

We could try $n = 2C$, but this may not be an integer, leave alone even. We modify this a little: let N be an integer with $N \geq C$.

Question: Why does this exist? We will turn to this problem later.

Now we take $n = 2N$ and observe that $n/2 = N \geq C$, which was all that remained.

Remark: One should resist the temptation to be more general than called for, so it is better to say take $n = 2N$ rather than saying *take n to be some even integer that is at least $2C$*. This is because, if we complicate the conditions n must satisfy, it becomes unclear whether some n satisfies all of them.

3. Derivative bounded below

We consider a problem similar to, but a little more complicated than, our first analysis problem.

3.1. **Problem:** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable real-valued function so that $f'(x) > 1$ for all $x \in \mathbb{R}$. Show that the function $f(x)$ is **not** bounded above.

3.2. **How to solve it:**

3.2.1. *Examining the problem.*

- Firstly, the *hypothesis* - what is given to you. Here this is not explicit but a condition on the function f. So it is worth exploring with some examples and our intution.
- The hypothesis says that the derivative is positive, so the function is increasing. It is useful to ask whether this is enough. Perhaps a function may increase at all times, but the rate of increase may get close to zero fast enough that the function does not become unbounded.
- Indeed this is the case. Consider the function $f(x) = \frac{e^x}{1+e^x}$. Observe that e^x is always positive, and so $0 < e^x < e^x + 1$. So $f(x)$ is bounded both above and below, as we get $0 < f(x) < 1$. On the other hand, we calculate that the derivative of $f(x)$ is always positive (do this).
- So it is not enough that the derivative is positive. But our hypothesis tells us that the derivative is not merely positive, but greater than 1. This means that $f(x)$ is not merely increasing everywhere, but doing so at speed greater than 1. Keep in mind that we must use this in our proof.
- Next, the *conclusion* we need is to show that the function $f(x)$ is **not bounded**.

3.2.2. *Recalling relevant definitions and results.*

- First, consider the conclsuion. For this, the one definition to recall is that of a *bounded function*. As we want to show that $f(x)$ is not bounded above, let us recall what it means for a_n to be bounded above.

" The function $f(x)$ is bounded above if there exists a real number C so that for all $x \in \mathbb{R}$, we have $f(x) < C$. "

- In this case, it turns out that we are best off working straight from the definition.
- Now turn to the hypothesis, namely $f(x)$ is a continuously differentiable function with $f'(x) > 0$. Most often, when working with a hypthesis involving the derivative of a function, we work not with definitions but with one of the important theorems concerning these: the **Mean Value theorem** (or its special case Rolle's theorem), the **Fundamental theorem of Calculus**, the result that *devative is zero at maxima and minima*, etc.
- In this case we will use the Mean Value theorem. There is an alternative approach using instead the Fundamental theorem of Calculus. There is no magic recipe telling us what to use. We learn from experience what is most likely to work, but we will still have to try different things till they work.
- Let us recall the mean value theorem. If $f : [a, b] \to \mathbb{R}$ is a continuously differentiable function, then there is some point c with $a < c < b$ so that
$$f(b) - f(a) = f'(c)(b - a).$$

3.2.3. *Strategy for proof.* Let us emphasise again that we often have to try different approaches till one succeeds. It is with experience that one learns which approach is most likely to succeed in a given situation, and, crucially, how long to follow up on an approach before changing it.

In this case, we again prove by contradiction. We want to show that the function $f(x)$ is not bounded, so we assume that it is and work from there.

Assume boundedness: By the definition we just recalled, this means that we assume the following:

"There is a constant $C > 0$ so that for all $x \in \mathbb{R}$, $f(x) < C$"

Remember C is fixed, but we do not know what it is. So we look for a contradiction treating C as a fixed but unknown constant. So our assumption is

"for all $x \in \mathbb{R}$, $f(x) < C$."

Let us see why this is impossible.Out inution says that as $f(x)$ is increasing, it is likely that if x is large enough, then $f(x) > C$. But we need to say precisely what large enough means, and show that $f(x) > C$ for such an x.

Let us give our large enough x a name, say $x = x_1$. Since we can expect x_1 to be large, let us assume that $x_1 > 0$. As mentioned earlier, it is frequently useful to invoke the mean value theorem.

But what should we take as a and b? The mean value theorem tells us how much the function grows in an interval, so in our situation we take the right endpoint to be $b = x_1$. There is no natural left end-point, so we may as well take $a = 0$.

In case you were mystified how we came up with $b = x_1$ and $a = 0$, let me emphasise again that there is no magic formula, and one will often come up with many useless substitutions and constructions before finding one that works. It is important to persist, as well as to look back over proofs once they are found and so get better at guessing the useful constructions and choices.

Another thing to reiterate, we seemed to pull 0 out of a hat, and you have probably guessed that there is nothing special about 0. But where any number works, it is often better to just pick one (typically some old friend like 0 or 1) rather than getting stuck like the proverbial donkey between two stacks of hay (who starved as there was no way to decide which one of the stacks to turn to.

Let us see now what the mean value theorem gives us, with $b = x_1$ and $a = 0$. We obtain that for some c with $0 < c < x_1$, we have $f(x_1) - f(0) = f'(c)(x_1 - 0)$.

We do not know what c is, but we do know that for all $x \in \mathbb{R}$, and in particular $x = c$, we have $f'(x) > 0$. We also have assumed $x_1 > 0$. So we can deduce that

$$f(x_1) - f(0) > 1(x_1 - 0),$$

In other words,

$$f(x_1) > f(0) + x_1.$$

This is just what the Doctor ordered - remember we have to show that f(x_1) is large if x_1 is large. What we have is an inequality saying that $f(x_1)$ is larger than the right hand side, namely $f(0) + x_1$. Let us examine this a little carefully.

The first term, $f(x)$, is something that our hypothesis tells nothing about. It may be a large negative number for all we know, perhaps $-10^{10^{100}}$. But we do know that it is *fixed*, in particular independent of x_1. So we need to choose x_1 so large that the sum $f(0) + x_1$ becomes as positive as we want.

To make all this precise, let us examine our goal. We want to show that it is not true that $f(x) < C$ for all $x \in \mathbb{R}$. We have already seen how to negate such statements, so we conclude that we have to show:

"there exists $x \in \mathbb{R}$ so that $f(x) \geq C$."

We have given such an x a name, $x = x_1$, but we need a recipe for actually choosing it. More precisely, we need to choose x_1 so that

$$f(x_1) \geq C.$$

But recall that we know

$$f(x_1) > f(0) + x_1$$

So it is enough to ensure that

$$f(0) + x_1 \geq C,$$

i.e.,

$$x_1 \geq C - f(0).$$

We take $x_1 = max(C - f(0), 1)$. This ensures x_1 is positive and satisfies the above equation). We are done.

Putting it together:

To clarify things, let us look at a sketch of the proof again.

We prove by contradiction. Suppose function $f(x)$ is bounded. Then there exists a number C so that $f(x) < C$ for all x. We shall pick $x = x_1$, with $x_1 > 0$ so that $f(x_1) \geq C$ to obtain a contradiction.

Using the hypothesis and the Mean Value theorem, we deduce that for $x_1 > 0$ we have $f(x_1) > f(0) + C$. So, if $x_1 = max(C - f(0), 1)$, we see that $f(x_1) \geq C$, giving the required contradiction.