Une construction de preuve en logique minimale

INF462 – Master S&T Informatique – Bordeaux 1

```
Coq < Variables P Q R : Prop.
P is assumed
Q is assumed
R is assumed
Coq < Lemma imp_perm : (P -> (Q -> R)) -> (Q -> (P -> R)).

1 subgoal

P : Prop
Q : Prop
R : Prop
============================
(P -> Q -> R) -> Q -> P -> R
Coq < Proof.

Coq <
```

$$\vdash (P \to (Q \to R)) \to (Q \to (P \to R))$$

```
Coq < Variables P Q R : Prop.
P is assumed
Q is assumed
R is assumed
Coq < Lemma imp_perm : (P -> (Q -> R)) -> (Q -> (P -> R)).

1 subgoal

P : Prop
Q : Prop
R : Prop
============================
(P -> Q -> R) -> Q -> P -> R
Coq < Proof.

Coq <   intro H.
```

$$\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))$$

```
Coq <   intro H.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
============================
Q -> P -> R
Coq <
```

$$\text{Lam}\ \frac{P \to (Q \to R) \vdash Q \to (P \to R)}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}$$

```
Coq <   intro H.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
============================
Q -> P -> R
Coq <   intro q.
```

$$\mathrm{Lam}\,\frac{P \to (Q \to R) \vdash Q \to (P \to R)}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}$$

```
Coq <   intro q.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
============================
P -> R
Coq <
```

$$
\text{Lam}\,\dfrac{\text{Lam}\,\dfrac{P \to (Q \to R), Q \vdash {\color{blue}P \to R}}{P \to (Q \to R) \vdash Q \to (P \to R)}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}
$$

```
Coq <   intro q.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
============================
P -> R
Coq <   intro p.
```

$$\text{Lam}\,\dfrac{\text{Lam}\,\dfrac{P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R}{P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)}}{\vdash (P \rightarrow (Q \rightarrow R)) \rightarrow (Q \rightarrow (P \rightarrow R))}$$

```
Coq <   intro p.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
p : P
============================
R
Coq <
```

$$
\dfrac{\dfrac{\dfrac{P \to (Q \to R), Q, P \vdash R}{P \to (Q \to R), Q \vdash P \to R}\,\text{Lam}}{P \to (Q \to R) \vdash Q \to (P \to R)}\,\text{Lam}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}\,\text{Lam}
$$

```
Coq <   intro p.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
p : P
============================
R
Coq <   apply H.
```

$$\text{Lam}\;\dfrac{\text{Lam}\;\dfrac{\text{Lam}\;\dfrac{P \to (Q \to R), Q, P \vdash R}{P \to (Q \to R), Q \vdash P \to R}}{P \to (Q \to R) \vdash Q \to (P \to R)}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}$$

```
Coq <   apply H.
2 subgoals

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
p : P
============================
P
subgoal 2 is:
Q
Coq <
```

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)}\,\mathrm{Var} \qquad P \to (Q \to R), Q, P \vdash \color{blue}{P}}{P \to (Q \to R), Q, P \vdash Q \to R}\,\mathrm{App} \qquad P \to (Q \to R), Q, P \vdash \color{blue}{Q}}{P \to (Q \to R), Q, P \vdash R}\,\mathrm{App}}{P \to (Q \to R), Q \vdash P \to R}\,\mathrm{Lam}}{P \to (Q \to R) \vdash Q \to (P \to R)}\,\mathrm{Lam}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}\,\mathrm{Lam}
$$

```
Coq <   apply H.
2 subgoals

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
p : P
============================
P
subgoal 2 is:
Q
Coq <   assumption.
```

$$
\dfrac{\dfrac{\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)}\,\text{Var} \qquad P \to (Q \to R), Q, P \vdash P}{P \to (Q \to R), Q, P \vdash Q \to R}\,\text{App} \qquad P \to (Q \to R), Q, P \vdash Q}{\dfrac{P \to (Q \to R), Q, P \vdash R}{\dfrac{P \to (Q \to R), Q \vdash P \to R}{\dfrac{P \to (Q \to R) \vdash Q \to (P \to R)}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}\,\text{Lam}}\,\text{Lam}}\,\text{Lam}}\,\text{App}
$$

```
Coq <   assumption.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
p : P
============================
Q
Coq <
```

$$
\text{Lam}\,\dfrac{\text{Lam}\,\dfrac{\text{Lam}\,\dfrac{\text{App}\,\dfrac{\text{App}\,\dfrac{\text{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)} \quad \text{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash P}}{P \to (Q \to R), Q, P \vdash Q \to R} \qquad P \to (Q \to R), Q, P \vdash \color{blue}{Q}}{P \to (Q \to R), Q, P \vdash R}}{P \to (Q \to R), Q \vdash P \to R}}{P \to (Q \to R) \vdash Q \to (P \to R)}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}
$$

```
Coq <   assumption.
1 subgoal

P : Prop
Q : Prop
R : Prop
H : P -> Q -> R
q : Q
p : P
============================
Q
Coq <   assumption.
```

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)}\,\text{Var} \quad \dfrac{}{P \to (Q \to R), Q, P \vdash P}\,\text{Var}}{P \to (Q \to R), Q, P \vdash Q \to R}\,\text{App} \qquad P \to (Q \to R), Q, P \vdash \color{blue}{Q}}{P \to (Q \to R), Q, P \vdash R}\,\text{App}}{P \to (Q \to R), Q \vdash P \to R}\,\text{Lam}}{\dfrac{P \to (Q \to R) \vdash Q \to (P \to R)}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}\,\text{Lam}}\,\text{Lam}
$$

```
Coq <   assumption.
Proof completed.
Coq <
```

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)}\,\text{Var} \quad \dfrac{}{P \to (Q \to R), Q, P \vdash P}\,\text{Var}}{P \to (Q \to R), Q, P \vdash Q \to R}\,\text{App} \quad \dfrac{}{P \to (Q \to R), Q, P \vdash Q}\,\text{Var}}{\dfrac{\dfrac{P \to (Q \to R), Q, P \vdash R}{P \to (Q \to R), Q \vdash P \to R}\,\text{Lam}}{P \to (Q \to R) \vdash Q \to (P \to R)}\,\text{Lam}}\,\text{App}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}\,\text{Lam}
$$

```
Coq <   assumption.
Proof completed.
Coq < Qed.
```

$$
\mathrm{Lam}\,\dfrac{\mathrm{Lam}\,\dfrac{\mathrm{Lam}\,\dfrac{\mathrm{App}\,\dfrac{\mathrm{App}\,\dfrac{\mathrm{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)} \quad \mathrm{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash P}}{P \to (Q \to R), Q, P \vdash Q \to R} \quad \mathrm{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash Q}}{P \to (Q \to R), Q, P \vdash R}}{P \to (Q \to R), Q \vdash P \to R}}{P \to (Q \to R) \vdash Q \to (P \to R)}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}
$$

```
Coq < Qed.
intro H.
intro q.
intro p.
apply H.
assumption.
assumption.
imp_perm is defined
Coq <
```

$$
\text{Lam}\,\dfrac{\text{Lam}\,\dfrac{\text{Lam}\,\dfrac{\text{App}\,\dfrac{\text{App}\,\dfrac{\text{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash P \to (Q \to R)} \quad \text{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash P}}{P \to (Q \to R), Q, P \vdash Q \to R} \quad \text{Var}\,\dfrac{}{P \to (Q \to R), Q, P \vdash Q}}{P \to (Q \to R), Q, P \vdash R}}{P \to (Q \to R), Q \vdash P \to R}}{P \to (Q \to R) \vdash Q \to (P \to R)}}{\vdash (P \to (Q \to R)) \to (Q \to (P \to R))}
$$