Population Modelling of Time Varying Vital Rates and Immigration*

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1 Introduction

The problem of understanding the growth/decline of human populations has attracted considerable interests in the past [9, pp.1–4]. As early as the seventeenth century, John Graunt (1620–74) studied the population of London in some detail and his study went as far as compiling a life table. The work of Thomas Malthus (1766–1834)—of the population principle fame [1]—in population analysis may be considered to some extent as the formulation of a mathematical model.

However the papers of Lotka [7], and Sharpe and Lotka [10] are usually taken as the beginning of a truly mathematical study of human populations.

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A seminal result of their analyses is the development of a mathematical model known as the ***Sharpe-Lotka single-sex deterministic population model***, henceforth referred to as the Sharpe-Lotka model. This model is widely used in the modelling and analysis of population growth, and is flexible to the extent that immigration may be incorporated into the basic model itself (see for example Gunadi [5]). As such, the Sharpe-Lotka model provides the basis of our present discussion and the reader is referred to Cerone [3] for a detailed account of the requisite theory as well as further study.

2 The Sharpe-Lotka model

2.1 Working assumptions

The Sharpe-Lotka model was originally formulated using various assumptions [2, pp.11–15] and the main ones shall now be described. First, Sharpe and Lotka assumed that the human population under consideration is closed to migration; thus no one enters the population and no one leaves it. Next, the vital rates—i.e. the birth and death rates—are age-specific and independent of time. Third, without loss of generality the study can be conducted on either the female or the male sex; here we will consider the female population only. Furthermore we may add that a tacit assumption of the Sharpe-Lotka model is that the original population is stable, by which we mean that it changes according to some smooth, well-behaved function such as an exponential function.

2.2 Mathematical formulation of the model

Let us now turn to a brief overview of the mathematical formulation of the basic Sharpe-Lotka model; our discussion in this and the next two sections draws heavily on Cerone [3]. For continuous time t, let $B(t)$ denote the total birth rate due to all mothers at that particular time instance. If we consider those mothers of age x alive at time t, then the birth rate of these mothers within the (minuscule) age interval $[x, x + dx]$ may be expressed as

$$B(t - x)\ell(x)m(x)\, dx \tag{1}$$

Notice that the ***survivor function*** $\ell(x)$ is the chance that a newborn daughter will survive up to age x; the ***maternity function*** $m(x)$ is the rate of giving birth to a daughter by a female of age x. Integrating expression (1) over all ages we obtain

$$B(t) = \int_0^\infty B(t-x)\phi(x)\,dx \tag{2}$$

which gives the total birth rate due to all mothers. In practice (and as a matter of common sense) we do not integrate from zero to positive infinity, but from zero to the oldest possible age ω, which is usually taken to be $\omega = 100$ years. However we assume that the child bearing ages are between 12.5 and 60 years old, hence the effective contribution of females below 12.5 and above 60 is zero. Comparing equations (1) and (2), the substitution $\phi(x) = \ell(x)m(x)$ is called the ***net maternity function*** since it provides the probability that a female lives up to age x and giving birth to a daughter within the immediate future.

Equation (2) may be expressed in the form of a Volterra integral equation of the second kind (see Moiseiwitsch [8]), namely

$$B(t) = F(t) + \int_0^t B(t-x)\phi(x)\,dx \tag{3}$$

where $F(t)$ is the birth rate at time t due to the parent population. Taking the Laplace transform of equation (3) and using the convolution theorem (refer to the discussion in Zill [11, pp.296, 321]), we obtain

$$B^*(p) = F^*(p) + \phi^*(p)B^*(p)$$

or alternatively

$$B^*(p) = \frac{F^*(p)}{1-\phi^*(p)} \tag{4}$$

An interesting point about the denominator $1-\phi^*(p)$ in equation (4) deserves mention. Taking the zero of $1-\phi^*(p)$, its dominant real root r plays a significant role in determining the asymptotic behaviour of the population under study. For the present discussion, this root leads to an eventually stable population in the sense discussed in subsection 2.1.

3 Time varying net maternity function

Up until now, our net maternity function $\phi(x)$ is a function in the single variable x, namely continuous age, and is completely independent of time. In order to render the Sharpe-Lotka model as a realistic model, if we introduce a time variable t into the net maternity function, then what we end up with is a total birth rate function of the form

$$B(t) = F(t) + \int_0^t B(t-x)\Phi(x,t)\,dx \tag{5}$$

Equation (5) is an extension of the Sharpe-Lotka model to the case of time-dependency in the net maternity function. It is essentially the same as equation (3), except that now we have a time dependent net maternity function $\Phi(x,t)$ instead of the simpler $\phi(x)$. Our function $F(t)$ becomes

$$F(t) = N(0)\int_0^\infty \frac{a(x,0)}{L(x,0)}\Phi(x+t,t)\,dx$$

with

$$N(t) = \int_0^\infty B(t-x)L(x,t)\,dx \tag{6}$$

and

$$A(x,t) = N(t)a(x,t) \tag{7}$$

where $a(x,t)$ is the time-dependent age density and $L(x,t)$ is the time-dependent survivor function (see Cerone [2]). Here $N(t)$ gives the population number, i.e. the total number of females at time t, and $A(x,t)$ is the age distribution. In theory the solutions of equations (5), (6) and (7) yield the total births, population numbers and age distribution respectively.

4 The Keyfitz problem

Recall from subsection 2.2 that $\phi(x)$ is the net maternity function of our original population. Denote by α and β the minimum and maximum child bearing ages, respectively. Integrating $\phi(x)$ from α to β we have

$$R = \int_\alpha^\beta \phi(x)\,dx$$

which is the expected number of children that a female will bear during her lifetime.

In this section we consider the net maternity function in some detail and discuss various effects that it exerts on our population as it changes to a replacement level maternity behaviour. The relevant theory may be found in Keyfitz [6]. Our discussion is best understood within the context of countries whose governing bodies are concerned about their respective "unbearably" large populations, and wish to implement policies designed to keep their population numbers under control. The example of the China one-child policy immediately springs to mind.

4.1 Abrupt transition to replacement level

Suppose that, as a result of government policy or otherwise, it is dictated that each member of our (female) population is permitted to bear only one child (a daughter) during her lifetime—i.e. a population with replacement only maternity behaviour. Further assume that we have a growing population. As the net maternity function changes from $\phi(x)$ to the replacement level net maternity function $\phi(x)/R$, how does the above policy affect our population?

Keyfitz [6] originally considered the case wherein the change to replacement level is an abrupt change, that is, the policy has immediate effect. Based upon empirical evidence, he showed that a growing population would continue to grow for a period of time before it approaches the level of asymptotic zero growth population. This phenomenon is referred to as the "momentum" of population growth (Keyfitz [6]).

4.2 Gradual transition to replacement level

In contrast to the abrupt change that Keyfitz considered (see Subsection 4.1), Cerone and Keane [4] and Cerone [2] considered a gradual transition to a replacement level net maternity function. In particular they studied asymptotic effects on the population, where the transition to a replacement level maternity behaviour is governed by some exponential function.

Let our net maternity function be defined by $\Phi(x,t) = \psi(t)\phi(x)$ with $\psi(0) = 1$ and suppose that $\lim_{t\longrightarrow\infty} \psi(t) = 1/R$. Assuming that $B(t) = Qe^{rt}$ where $r > 0$, then we obtain

$$B(t) = \psi(t)\left\{Q\int_0^{\infty} e^{-rx}\phi(x+t)\,dx + \int_0^t B(t-x)\phi(x)\,dx\right\} \tag{8}$$

and write

$$\psi(t) = \frac{1}{R} + \left(1 - \frac{1}{R}\right)e^{-\lambda t}, \qquad \lambda > 0 \tag{9}$$

The parameter λ in equation (9) may be interpreted as the rate of transition to replacement level maternity behaviour.

Using the Australian 1967 data, Cerone [2, 3] constructed Table 1. Going from left to right, the first column lists various values of λ, the column with header Q_2/Q lists asymptotic birth rates, the column after that lists the asymptotic population growth, and the last column lists relative percentage differences from the Keyfitz values (i.e. those values appearing in the second row). The lower is the value of λ, the more gradual is the transition to replacement level (refer to Figure 1). Consequently in the long run, the birth rate increases and the population number grows. For example, with the abrupt transition to replacement level maternity behaviour as studied by Keyfitz [6], the asymptotic birth rate is about 85% of the birth rate of the original population, while in the long run the original population would have grown by 25%.

λ	Q_2/Q	P_2/P	Relative % difference
∞	0.859355	1.25505	000.00
$10r$	0.953712	1.39286	010.98
$7r$	0.998378	1.45809	016.18
$4r$	1.119800	1.63543	030.31
r	2.503525	3.65630	191.54

Table 1: Asymptotic birth and population growth rates, as taken from Cerone [3, Table 6, p.23].

5 The vacation project

This was a six-week summer vacation project, undertaken during February and early March of 2006 under the supervision of Associate Professor Pietro Cerone and Dr Jakub Szajman. For background information, the vacation scholar familiarized himself with the area of demography (Cerone [2, 3], Gunadi [5] and Pollard [9]).

Although theoretical and asymptotic results existed for the basic Sharpe-Lotka model and its various extended forms, a major objective of the project was the numerical solution of equations (5), (6) and (7) to obtain the transient behaviour of a population model with time dependent net maternity

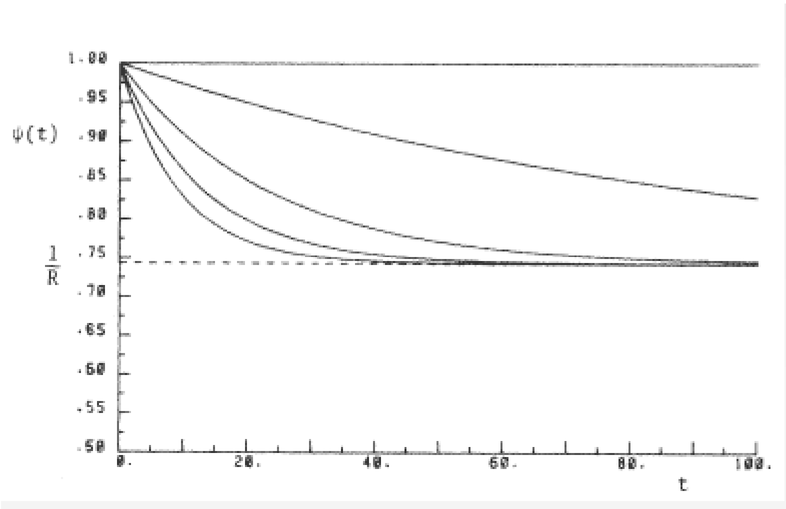

Figure 1: Graphs of $\psi(t)$ as given in equation (9) for varying λ. From top to bottom the graphs represent $\psi(t)$ for $\lambda = 0$, r, $4r$, $7r$, $10r$ and the broken line is the abrupt Keyfitz change to replacement level (corresponding to allowing $\lambda \longrightarrow \infty$ in equation (9)). Figure taken from Cerone [3, Figure 1, p.16].

function. Thus at the outset the scholar was required to learn the FORTRAN programming language, especially the FORTRAN 77 standard, and he investigated the operation of an existing FORTRAN programme written over thirty years ago by Associate Professor Cerone. In these tasks, technical assistance was rendered by Dr Szajman.

Investigating the operation of the abovementioned FORTRAN programme was only possible once all of its components had been appropriately assembled. At the outset, a few components of the programme were readily available as electronic text files. Those which were not available in electronic form were converted to such format. Thus the scholar found himself typing these various programme components from a photocopied version of the original source code printout. Unfortunately as with most cases in which data are manually entered into a computer, errors of various sorts were inadvertently introduced into the text files containing the newly typed code. As a consequence the first two weeks of the project were spent debugging the overall programme, comparing line by line the vast majority of code with its counterpart on paper, and testing for accuracy any newly debugged version of the programme.

The first half of the third week was spent preparing for an oral presentation to be delivered at the Lindfield branch of CSIRO, over the time period 16^{th} to 17^{th} February 2006 (the "Big Day In!" convention). The content of this presentation was essentially the material presented in the first four sections of this paper. During the fourth week, a small portion of the programme was slightly modified. In particular the function $\psi(t)$ as defined in equation (9) was coded into the programme itself; the resulting modified version was tested for accuracy of programming logic as well as accuracy of output. In the final two weeks, an attempt was made to numerically determine the transient population numbers and age distribution (see equations (6) and (7)).

6 Future work

That which was accomplished during this project serves to provide a foundation for future on-going work in the modelling of transient population dynamics under changing vital rates and immigration. There exist mathematical models that take immigration into account and predict long term or asymptotic behaviours of such models (see for example Cerone [3] and Gunadi [5]). However in some circumstances it is desirable to have at hand certain vital behaviours of a population within the immediate future so that, for example, social planners can draw upon such information when it comes to drafting policies concerning human services and urban planning. One of our objectives for future work is to develop/assemble a suite, albeit specialized, of FORTRAN programmes that provides transient behaviours of various human population models.

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