Population Modelling of Time Varying Vital Rates and Immigration

Minh Van Nguyen
ICE-EM AMSI Vacation Scholar
School of Computer Science and Mathematics
http://csm.vu.edu.au

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Preface

Modelling of the human population has attracted considerable interest in the past but never more so than currently since it underpins many of the social and economic considerations facing us. The current work looks at developing the numerical solution of the effect of time-varying maternity behaviour and immigration. The population, unlike in some previous studies, is not assumed to be initially stable. The Sharpe-Lotka single sex deterministic population model is used as the basis for the analysis.

Please refer to http://rgmia.vu.edu.au/cerone/pc.pdf for further details.

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1. Structure of Presentation

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Abstract

The resultant behaviour of a population in response to changes of the age-specific birth and death rates with time is investigated.

The deterministic one-sex population model of Sharpe and Lotka is used as the basis for the analysis. The asymptotic behaviour is determined for a population with a time dependent net maternity function.

The problem of Keyfitz on the momentum of population growth is generalised to contain a gradual exponential scaling (at a rate λ) of the age-specific rate to the level of bare replacement.

Models are also proposed which allow for a time dependent change of the initial net maternity functions more general than the simple exponential. The asymptotic behaviour of the ensuing populations may then be evaluated.

1. Transient and Asymptotic Solution of the Sharpe-Lotka Model – The Basic Model

Denote by $B(t)$ the total birth rate at time t due to all mothers, $m(x)$ =maternity function, $\ell(x)$ =survival function (fraction of newborns surviving to age x) then the birth rate of mothers alive at time t, of age x to $x+dx$, is

$$B(t-x)\,\ell(x)\,m(x)\,dx,$$

and so integrating (summing) over all ages,

$$B(t) = \int_0^\infty B(t-x)\,\phi(x)\,dx. \tag{1}$$

It can be seen that (1) may be written in the form of a Volterra integral equation of the second kind with a difference kernel, namely,

$$B(t) = F(t) + \int_0^t B(t-x)\,\phi(x)\,dx \tag{2}$$

where d =stable age density,

$$F(t) = \int_t^\infty B(t-x)\,\phi(x)\,dx = \int_0^\infty B(-x)\,\phi(x+t)\,dx, \tag{3}$$

is the birth rate at time t due to the females already alive at the origin, that is, due to the parent or initial population.

$$B(-x) = \frac{N(0)\,a(x,0)}{\ell(x)} = Qe^{-rx},$$

and hence, from (3),

$$F(t) = N(0)\int_0^\infty \frac{a(x,0)}{\ell(x)}\phi(x+t)\,dx. \tag{4}$$

Formally taking the Laplace transform of (2) we obtain, upon using the *convolution theorem*,

$$B^* (p) = F^* (p) + \phi^* (p) B^* (p)$$

or

$$B^* (p) = \frac{F^* (p)}{1 - \phi^* (p)}, \tag{5}$$

where $*$ denotes the one-sided Laplace transform viz.

$$u^* (p) = \int_0^{\infty} e^{-pt} u (t) \, dt.$$

Equation (5) may be inverted using standard residue theory. The dominant root of r of $\phi^*(p) = 1$ leads to an eventually stable population.

2. Time Varying Net Maternity Function

2.1. The Current Project

For time varying net maternity functions, the following relationships result for the Sharpe-Lotka model:

$$B(t) = F(t) + \int_0^t B(t-x)\,\Phi(x,t)\,dx, \tag{6}$$

where

$$F(t) = N(0)\int_0^\infty \frac{a(x,0)}{L(x,0)}\Phi(x+t,t)\,dx \tag{7}$$

$$B(-x) = N(0)\frac{a(x,0)}{L(x,0)} \tag{8}$$

$$N(t) = \int_0^\infty B(t-x)\,L(x,t)\,dx \tag{9}$$

and

$$\begin{aligned} A(x,t) &= N(t)\,a(x,t) \qquad (10)\\ &= N(0)\frac{a(x-t,0)}{L(x-t,0)}L(x,t)\,H(x-t) + B(t-x)\,L(x,t)\,H(t-x) \end{aligned}$$

$$B(-x) = Be^{-rx}, \quad \text{initially stable population,} \tag{11}$$

with $H(u) = 1,\ u > 0$ and $H(u) = 0,\ u \le 0$ (Heaviside).

The solution of (6), (9) and (10) give the transient total births $B(t)$, population numbers $N(t)$ and age distribution $A(x,t)$ respectively.

• One of the main aims of this project is the numerical solution of these to obtain the transient behaviour.

(7) was solved numerically (following the procedures of Linz (1969), DeBoor (1971) and, Campbell and Day (1971)), where the initial population has **not** been assumed to be initially stable.

$\phi(x)$, $A(x,0)$ and $\ell(x)$ are provided as discrete data and spline interpolation is used.
We note that $\phi(x) \neq 0$, $\alpha < x < \beta$ and $A(x,0)$, $\ell(x) \neq 0$, $0 < x < w$.

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2.2. Asymptotic Effects and Population Momentum

Keyfitz (1971b)

$$\Phi(x,t) = \phi(x) H(-t) + \frac{\phi(x)}{R} H(t) \quad - \quad \text{abrupt change at } t = 0.$$

For an initially stable population,

$$B(t) = Qe^{rt}, \quad N(t) = Pe^{rt}, \quad r > 0$$

$$B(t) \backsim Q_2 = Q\frac{R-1}{rR\kappa}, \quad N(t) \backsim P_2 = Pe_0 b \cdot \frac{R-1}{rR\kappa} \tag{12}$$

where

$$\kappa = \frac{\int_0^\infty x\phi(x)\,dx}{\int_0^\infty \phi(x)\,dx} = \frac{M_1}{M_0}, \quad M_j = \int_0^\infty x^j \phi(x)\,dx, \quad M_0 = R,$$

$$P = Q\int_0^\infty e^{-rx} l(x)\,dx, \quad P_2 = Q_2 \int_0^\infty l(x)\,dx$$

$$\frac{P_2}{P} = \frac{Q_2}{Q} \cdot \frac{\int_0^\infty l(x)\,dx}{\int_0^\infty e^{-rx} l(x)\,dx}$$

$$b = \frac{Q}{P} = \frac{1}{\int_0^\infty e^{-rx} l(x)\,dx} \quad \text{– crude birth rate (in initially) stable population}$$

$$\mathring{e}_0 = \int_0^\infty l(x)\,dx \quad \left(= \frac{1}{b_2} = \frac{P_2}{Q_2} \right) \quad \text{– life expectancy at birth}$$

• Using a number of examples, Keyfitz evaluated (12) and showed that a growing population would continue to grow even under an abrupt change to replacement maternity behaviour.

2.3. Asymptotic Effects of an Exponential Transition to Replacement (Cerone and Keane (1978a), Cerone (1979))

For $\Phi(x,t) = \psi(t)\phi(x) = M(x,t)\ell(x)$ with $\psi(0) = 1$ and $\lim_{t\to\infty}\psi(t) = \frac{1}{R}$ we have assuming $B(t) = Qe^{rt}$, $r > 0$ then

$$B(t) = \psi(t)\left\{Q\int_0^\infty e^{-rx}\phi(x+t)\,dx + \int_0^t B(t-x)\phi(x)\,dx\right\}. \tag{13}$$

Let

$$\psi(t) = \frac{1}{R} + \left(1 - \frac{1}{R}\right)e^{-\lambda t}, \quad \lambda > 0. \tag{14}$$

- A method was developed for determining the stationary asymptotic value of $B(t)$ (see [?]). An **upper bound** for Q_2 given by

$$Q_2 < Q\frac{R-1}{rR\kappa}\cdot\frac{\lambda}{\lambda - r}, \quad \lambda > r > 0. \tag{15}$$

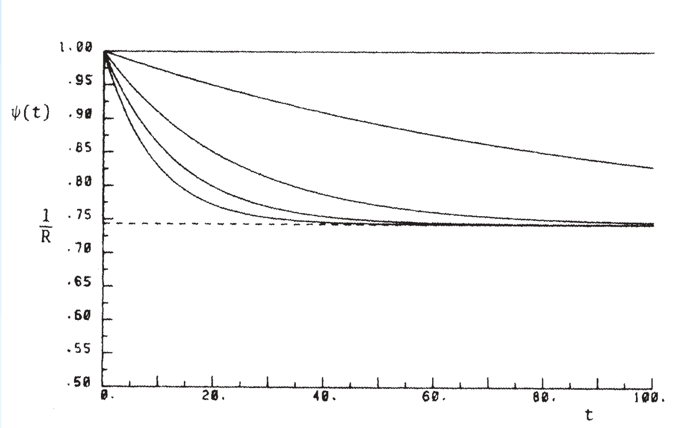

Figure 1: Diagram showing $\psi(t)$ as given by (14), versus time. From top to bottom the graphs represent $\psi(t)$ for $\lambda = 0, r, 4r, 7r, 10r$, and the broken line is the abrupt Keyfitz change to replacement level (corresponding to allowing $\lambda \to \infty$ in (14). R and r are the values for the Australian Female's data, 1967.

λ	Q_2/Q	P_2/P	Relative % Difference From the Keyfitz values
∞	0.859355	1.25505	0
$10r$	0.953712	1.39286	10.98
$7r$	0.998378	1.45809	16.18
$4r$	1.119800	1.63543	30.31
r	2.503525	3.65630	191.54

Table 1: Asymptotic birth rate Q_2 for the model (13) – (14). The asymptotic total number P_2 is also given, and the values are compared to those resulting from an abrupt Keyfitz change ($\lambda \to \infty$) for the Australian 1967 data.

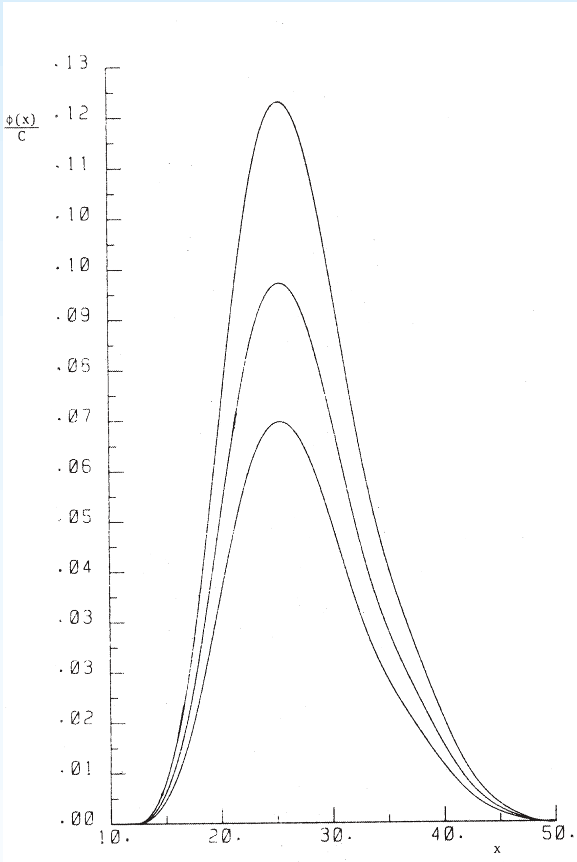

Figure 2: Diagram of $\phi(x)/C$ where $\phi(x)$ is the net maternity function of 1967 Australian females. From top to bottom, $C = 0.75, 1.0, R$.

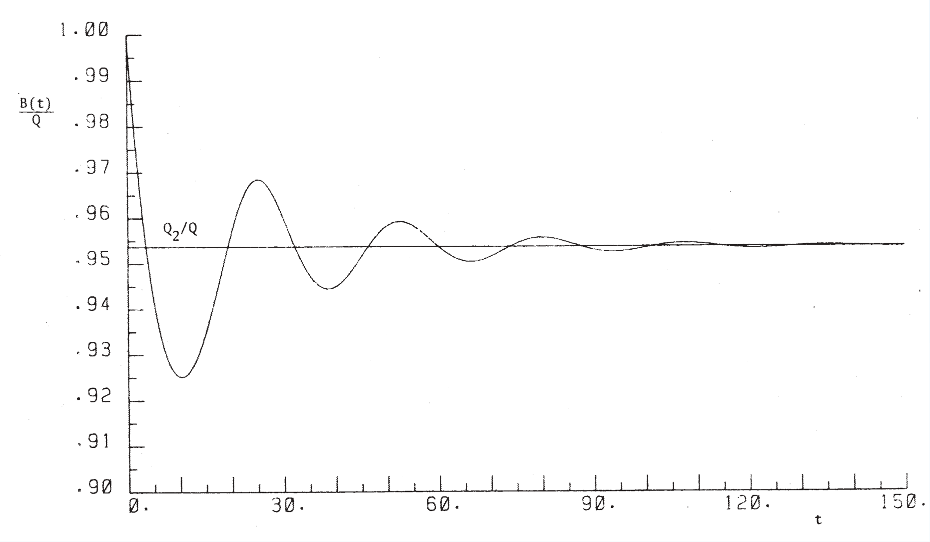

Figure 3: $B(t)$ resulting from an exponential change of the net maternity function from $\phi(x)$ to $\phi(x)/R$ at a rate $\lambda = 10r$. $\phi(x)$ is given in Figure 2 with $C = 1$. The graph is that of $B(t)/Q$ versus time t. The straight line represents the asymptotic value Q_2/Q.

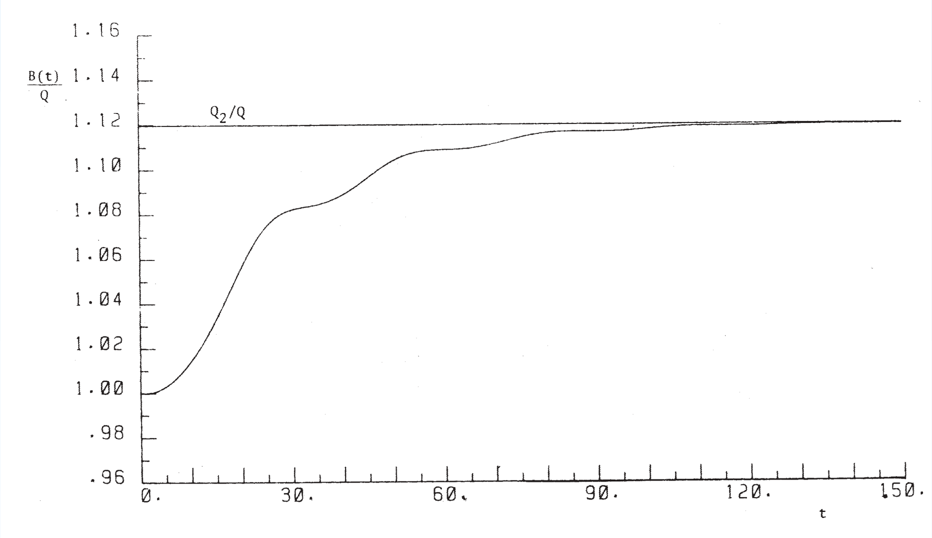

Figure 4: $B(t)$ resulting from an exponential change of the net maternity function from $\phi(x)$ to $\phi(x)/R$ at a rate $\lambda = 4r$. $\phi(x)$ is given in Figure 2 with $C = 1$. $B(t)$ approaches the asymptotic value Q_2/Q (the straight line).

3. Time Variation which does affect the Convolution

- Time Dependence for Parent Population Only

Cerone and Tognetti (1982) developed models to allow gradual changes in the reproductive behaviour which does not affect the convolution.

- Time Dependence no Greater than the Lowest Age of Childbearing, α

The effects of a general time dependent scaling of the net maternity function was considered for $\tau \leq \alpha$ such that

$$\Phi(x,t) = \begin{cases} \chi(t)\,\phi(t) & \text{for } \; 0 < t < \tau \leq \alpha, \\ \phi_2(x) & \text{for } \; t > \tau. \end{cases} \tag{16}$$

4. Transient Solution for the Exponential Models (Cerone (1996))

• Extension of model by considering the contribution from the complex roots of $\phi^*(p) = 1$ other than the dominant real root r,

• Time Dependence as a Sum of Exponentials

The techniques developed for the exponential time variation could be extended to treat more general changes described by sums of exponentials. So that for a change from $\phi_1(x)$ to $\phi_2(x)$

$$\Phi(x,t) = \phi_2(x) + \xi(t)\left[\phi_1(x) - \phi_2(x)\right], \tag{17}$$

where $\xi(0) = 1$ and $\lim_{t\to\infty} = 0$ we have specifically

$$\xi(t) = \sum_{m=1}^{M} \gamma_m e^{-\lambda_m t},$$

$\lambda_m > 0$ for all $m = 1, 2, \ldots, M$ and

$$\sum_{m=1}^{M} \gamma_m = 1.$$

Part B: Time Varying (Im)migration

Abstract

Stable population theory is extended to allow models of a constant stream of immigrants and also models with time-varying immigrant behaviour. Traditional methods of Laplace transforms have been applied to solve the Volterra single-sex integral population model for the total birth rate. Although the main emphasis was in developing expressions for the asymptotic effects of some simple but versatile models of time varying immigrant behaviour the transient, a solution may also be procured utilising the techniques in Part A obtained for closed populations.

5. Stable Population Theory with Immigration (Cerone, 1987 or 1993)

Espenshade, Bouvier, and Arthur (1982) examined what would happen to a population of below-replacement fixed fertility behaviour together with a constant number and age distribution of immigrants persisted indefinitely. They showed that an eventual stationary population that does not depend on the initial population would result.

Mitra (1983) removed the restriction of below replacement reproduction behaviour.

The problem of a constant flow of immigrants was also examined earlier by Coale (1973). Coale worked in the reverse direction by determining the fertility behaviour needed to result in an eventual stationary population, given a constant stream of immigrants. Coale also concluded, like Mitra, that if replacement level vital rates persist, the eventual population behaviour will be linear. Coale made the same assumption as Mitra in trying to determine one of the parameters.

A general migration model was developed by Langhar (1972). Cerone (1987) used the Sharpe-Lotka single-sex population model to determine the eventual population behaviour under a constant stream of immigration using the conventional method of Laplace transforms.

5.1. The Model with Constant Immigration

If we let $I(x)$ be the number of females immigrating at age x, $m(x)$ the rate of bearing daughters, and $l(x)$ the fraction of newborn females who will survive to age x, then the number of births at time t is given by (Espenshade, Bouvier, and Arthur, 1982)

$$B(t) = \int_0^\infty A(x,t)\, m(x)\, dx, \quad t \geq 0, \tag{18}$$

where

$$A(x,t) = B(t-x)\, l(x) + S(x)\, l(x) \tag{19}$$

with $S(x) = \int_0^x [I(u)/l(u)]\, du$.

Thus,

$$B(t) = G(t) + \int_0^t [B(t-x) + S(x)]\, \phi(x)\, dx,$$

where

$$G(t) = \int_t^\infty [B(t-x) + S(x)]\,\phi(x)\,dx$$

with $\phi(x) = m(x)\,l(x)$. To obtain an explicit expression for $G(t)$, where $t \le x$, we need to know the number of births *before* the chosen origin $t = 0$. However, a change of variable gives

$$G(t) = \int_0^\infty [B(-x) + S(x+t)]\,\phi(x+t)\,dx,$$

requiring only the number of births at $t = 0$, namely $B(-x)$. From equation (19) an expression of $B(-x)$ is readily obtained as

$$B(-x) = \frac{A(x,0)}{l(x)} - S(x),$$

so

$$\begin{aligned} G(t) &= \int_0^\infty \left[\frac{A(x,0)}{l(x)} - S(x) + S(x+t)\right]\phi(x+t)\,dx \\ &= \int_t^\infty \left[\frac{A(x-t,0)}{l(x-t)} + S(x) - S(x-t)\right]\phi(x)\,dx. \end{aligned}$$

Thus the equation for $B(t)$ may be written in the form

$$B(t) = b_I + F(t) + \int_0^t B(t-x)\,\phi(x)\,dx, \tag{20}$$

where $b_t = \int_0^\infty S(x)\,(x)\,dx$ and

$$\begin{aligned} F(t) &= \int_0^\infty \left[\frac{A(x,0)}{l(x)} - S(x)\right]\phi(x+t)\,dx \\ &= \int_t^\infty \left[\frac{A(x-t,0)}{l(x-t)} - S(x-t)\right]\phi(x)\,dx. \end{aligned} \tag{21}$$

We note that the immigrant contribution to the births comes through *both* the b_I and $F(t)$ terms.

Further note that equation (20) is similar in form to (6 and so a similar approach may be followed.

6. Stable Population Theory with Time Varying Immigration (Cerone and Gunadi (1999))

The effect of immigration has become of some importance. Originally, stable population theory was developed to study so called closed populations (Lotka (1939), Sharpe and Lotka (1911)). Over the past quarter of a century or so open populations have been investigated and the effect of immigration on the long-run behaviour has been examined.

Keyfitz (1971) investigated the long-run effect of emigration on a population while Coale (1972) considered the reduction in fertility needed to counterbalance the effect of a steady stream of immigrants. Epenshade *et al.* (1982) and Mitra (1983) analysed the consequences of a constant indefinite stream of immigrants while Cerone (1986) extended stable population theory to include a constant stream of immigrants.

Mitra (1990) examined the vital rates and the age structure of a long-term stationary population brought about by the effect of a constant stream of immigrants on below replacement fertility regimen of the local population. Smertmann (1990) investigated the *rejuvenating* force reflecting through the eventual age - structure of the ensuing population under similar conditions as Mitra (1990). Blanchet (1989) examines the possibility of regulating the age - structure of an ensuing population under a constant stream of immigrants.

General immigration models have been developed in the past. Sivamurthy (1982) used a discrete formulation of a Leslie matrix and the population was projected. An integral equation model was developed by Langhaar (1972) which may be solved numerically. In this article however, some simple models shall be developed in order to investigate the effect of time variation of immigration levels. Further, a number of authors including Feichtinger and Steinmann (1992) and Friedlander and Feldmann (1993), have indicated that the adoption of local fertility behaviour by the immigrant population to be unrealistic. A model is developed and analysed that allows for a gradual transition of the fertility behaviour of an immigrant population to that of the local population.

7. Model Development

A general integral equation model was developed by Langhaar (1972) which includes time varying migration and vital rates. He was able to solve equations numerically. We shall only consider the female portion of the population as is customary. If we let $m(x,t)$ be the expected number of female offspring, to a woman aged x at time t, over the interval $(x, x+dx)$ then the number of female births $B(t)$ is given by (Langaar (1972))

$$B\left(t\right)=\int_{0}^{\infty}A\left(x,t\right)m\left(x,t\right)dx,\qquad t\geq 0 \tag{22}$$

where

$$A\left(x,t\right)=B\left(t-x\right)l\left(x,t\right)+J\left(x,t\right), \tag{23}$$

is the age-distribution, $l(x,t)$ is the time dependent survivor function and

$$J\left(x,t\right)=\psi\left(x,t\right)l\left(x,t\right), \tag{24}$$

with $J(x,t)dx$ being the number of first generation immigrants at time t within the age group $(x, x+dx)$.

It should further be highlighted that equation (23) holds for $t \geqslant x$ and there is a lack of information about the population prior to $t=0$, our chosen origin. Following Cerone (1987) from equations (22) and (23) (and using (24)) gives

$$B\left(t\right)=G\left(t\right)+F\left(t\right)+\int_{0}^{t}B\left(t-x\right)\Phi\left(x,t\right)dx \tag{25}$$

where

$$G\left(t\right)=\int_{0}^{\infty}J\left(x,t\right)m\left(x,t\right)dx=\int_{0}^{\infty}\psi\left(x,t\right)\Phi\left(x,t\right)dx \tag{26}$$

and

$$F\left(t\right)=\int_{t}^{\infty}B\left(t-x\right)\Phi\left(x,t\right)dx=\int_{0}^{\infty}B\left(-x\right)\Phi\left(x+t,t\right)dx.$$

Hence from (23)

$$F(t) = \int_0^\infty \left[\frac{A(x,0) - J(x,0)}{l(x,0)}\right] \Phi(x+t,t)\, dx$$

or

$$F(t) = \int_t^\infty \left[\frac{A(x-t,0) - J(x-t,0)}{l(x-t,0)}\right] \Phi(x,t)\, dx. \tag{27}$$

The total population numbers $N(t)$ may be obtained by simply summing the combined age-distribution over all ages to give from (23)

$$N(t) = \int_0^\infty B(x,t)\, l(x,t)\, dx + \int_0^\infty J(x,t)\, dx.$$

Further, the age-density may also be determined as the ratio of the age-distribution over the total population, viz.

$$a(x,t) = \frac{A(x,t)}{N(t)}.$$

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