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Brief Paper

Homogeneous Lyapunov functions for systems with structured uncertainties☆

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Abstract

The problem addressed in this paper is the construction of homogeneous polynomial Lyapunov functions (HPLFs) for linear systems with time-varying structured uncertainties. A sufficient condition for the existence of an HPLF of given degree is formulated in terms of a linear matrix inequalities (LMI) feasibility problem. This condition turns out to be also necessary in some cases depending on the dimension of the system and the degree of the Lyapunov function. The maximum ℓ_∞ norm of the parametric uncertainty for which there exists a homogeneous polynomial Lyapunov function is computed by solving a generalized eigenvalue problem. The construction of such Lyapunov functions is efficiently performed by means of popular convex optimization tools for the solution of problems in LMI form. Comparisons with other classes of Lyapunov functions through numerical examples taken from the literature show that HPLFs are a powerful tool for robustness analysis.

Keywords: Robustness; Lyapunov function; Homogeneous forms; Linear matrix inequalities (LMI)

1. Introduction

Lyapunov functions are a standard tool for tackling robust analysis of linear systems affected by time-varying structured uncertainties. Quadratic Lyapunov functions have been considered by many authors, since a long time (see e.g. Šiljak, 1969; Narendra & Taylor, 1973; Zhou & Khargonekar, 1987). It is, however, widely recognized that quadratic functions lead to conservative estimates of the robust stability margin. Hence, nonquadratic Lyapunov functions have been addressed in the literature. Piecewise quadratic Lyapunov functions have been considered for both linear systems with time-varying perturbations (Xie, Shishkin, & Fu, 1997) and switching linear systems (Johansson & Rantzer, 1998). Polyhedral Lyapunov functions have been introduced in Brayton and Tong (1979) and successively considered by several authors. It has been shown that they are not conservative for robust analysis and synthesis (Blanchini, 1995) of linear systems with time-varying structured uncertainties. The main drawback of polyhedral functions is that the computational burden required by their construction dramatically increases with the dimension of the system and the number of vertices of the polytope of matrices describing the uncertainty.

Homogeneous polynomial Lyapunov functions (HPLFs) are a viable alternative to the above classes of Lyapunov functions. The fact that this class of Lyapunov functions can improve robust stability results provided by quadratic Lyapunov functions has been recognized since long time (Brockett, 1973). In Zelentsovsky (1994), the use of HPLFs to prove robust stability of linear systems with time-varying uncertainties has been considered and an approach based on the S-procedure has been proposed to enhance the robustness degree. More recently, it has been shown that for these systems robust stability is equivalent to the existence of a smooth Lyapunov function that turns out to be the sum of squares of homogeneous polynomial forms (Blanchini & Miani, 1999). This suggests that the computation of HPLFs can be pursued via the numerical tools in (Chesi, Tesi, Vicino, & Genesio, 1999; Parrilo, 2000a, b; Chesi, Garulli,

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Tesi, & Vicino, 2003), which rely on the convexity property of sum of squares of homogeneous forms.

In this paper, the problem of constructing HPLFs is tackled by means of convex optimization techniques based on linear matrix inequalities (LMI) (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Nesterov & Nemirovsky, 1993) and on certain properties of homogeneous forms which lead to a suitable matricial representation (Chesi, Tesi, Vicino, & Genesio, 2001; Chesi et al., 2003). The aim of the paper is twofold: to formulate conditions for the existence of HPLFs of given degree in terms of LMI feasibility problems, and to address the problem of computing the maximum ℓ_∞ norm of the parametric uncertainty for which an HPLF exists. Several examples are presented, which demonstrate that HPLFs are a powerful tool for robustness analysis.

The paper is organized as follows. Section 2 contains the problem formulation and preliminary material on homogeneous forms. The existence conditions for HPLFs are provided in Section 3, where relationships with previous work are also discussed. Maximization of the ℓ_∞ norm of the uncertainty is treated in Section 4, while the computation of the HPLF achieving the optimal transient performance is addressed in Section 5. Several numerical examples are presented in Section 6 and some concluding remarks are given in Section 7.

2. Problem formulation and preliminaries

Let us consider the uncertain linear system

$$\dot{x}(t) = A(w(t))x(t), \tag{1}$$

where

$$A(w(t)) = \left(A_0 + \sum_{i=1}^{s} w_i(t) A_i \right) \tag{2}$$

and $A_0, \dots, A_s \in \mathbb{R}^{n\times n}$ are given matrices. The uncertain time-varying parameter vector $w(t) = (w_1(t), \dots, w_s(t))'$ is assumed to satisfy for all $t \geqslant 0$ the constraint

$$w(t) \in \mathscr{W} \triangleq \mathrm{co}\{w^1, \dots, w^r\}, \tag{3}$$

where $w^i \in \mathbb{R}^s$, $i = 1, \dots, r$, are given vectors and $\mathrm{co}\{\cdot\}$ denotes the convex hull.

The basic problem addressed in this paper is the construction of a Lyapunov function proving global asymptotic stability of system (1)–(3). The attention is restricted to a special class of Lyapunov functions: the homogeneous polynomial forms of degree $2m$.

Before proceeding, let us recall that a function $f_m(x)$ is a *homogeneous form* of degree m in $x \in \mathbb{R}^n$ if

$$f_m(x) = \sum_{i_1+i_2+\cdots+i_n=m} c_{i_1,i_2,\dots,i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n},$$

where $i_1, i_2, \dots, i_n$ are nonnegative integers, and $c_{i_1,i_2,\dots,i_n} \in \mathbb{R}$ are weighting coefficients. The form $f_m(x)$ is said *positive* if $f_m(x) > 0\ \forall x \neq 0$, and *nonnegative* if $f_m(x) \geqslant 0\ \forall x$.

Hereafter, the aim will be to find an HPLF of degree $2m$, denoted by $v_{2m}(x)$, such that:

(i) $v_{2m}(x) > 0$ for all $x \neq 0$;
(ii) $\dot{v}_{2m}(x) < 0$ for all $x \neq 0$ and for all $w(t) \in \mathscr{W}$.

We introduce a representation of homogeneous forms that will be exploited throughout the paper. Let $g_{2m}(x)$ be a homogeneous form of degree $2m$. Then, the *square matricial representation* (*SMR*) of $g_{2m}(x)$ is defined as $g_{2m}(x) = x^{\{m\}\prime} C_g x^{\{m\}}$, where $x^{\{m\}} \in \mathbb{R}^d$ is the base vector of homogeneous forms of degree m in x (containing all monomials of degree m), and $C_g = C_g' \in \mathbb{R}^{d\times d}$ is a coefficient matrix. It is not difficult to check that the dimension d of $x^{\{m\}}$ is given by $d = (n+m-1)!/(n-1)!m!$. An important property of the SMR is that matrix C_g is not unique. Indeed, if one considers the set of matrices $\mathscr{L} = \{L = L' \in \mathbb{R}^{d\times d}: x^{\{m\}\prime} L x^{\{m\}} = 0\ \forall x \in \mathbb{R}^n\}$, then the family of matrices C_g describing $g_{2m}(x)$ can be parameterized affinely as $C_g(\alpha) = C_g + L(\alpha)$, where $\alpha \in \mathbb{R}^{d_{\mathscr{L}}}$ is a vector of free parameters and $L: \mathbb{R}^{d_{\mathscr{L}}} \to \mathscr{L}$ is a linear parameterization of $\mathscr{L}$. In Chesi et al. (2003) it is shown that $\mathscr{L}$ is a linear space whose dimension is given by $d_{\mathscr{L}} = \frac{1}{2} d(d+1) - (n+2m-1)!/(n-1)!(2m)!$; moreover, details can be found in the same reference on how to generate the complete parameterization $L(\alpha)$. Hereafter, the representation

$$g_{2m}(x) = x^{\{m\}\prime} C_g(\alpha) x^{\{m\}} \tag{4}$$

with $x^{\{m\}} \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^{d_{\mathscr{L}}}$, will be addressed as the complete SMR (CSMR) of $g_{2m}(x)$.

3. Existence conditions for homogeneous Lyapunov functions

In this section, conditions for the existence of an HPLF for system (1)–(3) are provided. Such conditions are based on the CSMR of homogeneous forms previously introduced and are expressed as LMIs.

3.1. Sufficient condition

For a generic system $\dot{x}(t) = Ax(t)$, let us introduce the *extended matrix* $A_{\{m\}} \in \mathbb{R}^{d\times d}$, defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} x^{\{m\}} = \frac{\partial x^{\{m\}}}{\partial x} Ax = A_{\{m\}} x^{\{m\}}. \tag{5}$$

Extended matrices play a key role in the so-called power transformation, which has been extensively used in the analysis of control systems (Brockett, 1973; Barkin & Zelentsovsky, 1982, 1983). The following useful properties of the extended matrix $A_{\{m\}}$ hold (Brockett, 1973; Zelentsovsky, 1994):

(I) Let $A, B \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha A + \beta B)_{\{m\}} = \alpha A_{\{m\}} + \beta B_{\{m\}}.$$

(II) Let $A_{0,\{m\}}$ and $A_{i,\{m\}}$ denote, respectively, the extended matrices of A_0 and A_i, according to (5). Consider the *extended system*

$$\dot{x}^{\{m\}}(t) = \left(A_{0,\{m\}} + \sum_{i=1}^{s} w_i(t) A_{i,\{m\}}\right) x^{\{m\}}(t) = A(w(t))_{\{m\}} x^{\{m\}}(t).$$

Then, for any $v_{2m}(x) = x^{\{m\}\prime} V x^{\{m\}}$, one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} v_{2m}(x)\right|_{\dot{x}^{\{m\}}=A(w(t))_{\{m\}}x^{\{m\}}} = \left.\frac{\mathrm{d}}{\mathrm{d}t} v_{2m}(x)\right|_{\dot{x}=A(w(t))x}.$$

The following result provides a sufficient condition for establishing the existence of an HPLF of degree $2m$ for system (1)–(3).

Theorem 1. *Let $\bar{A}_j = A(w^j)$, $j=1,\ldots,r$ and let $\bar{A}_{j,\{m\}}$ denote the extended matrix of $\bar{A}_j$. If the system of LMIs*

$$V > 0,$$
$$-V\bar{A}_{j,\{m\}} - \bar{A}'_{j,\{m\}}V - L(\alpha^j) > 0, \quad j = 1,\ldots,r \tag{6}$$

admits a feasible solution $V = V' \in \mathbb{R}^{d\times d}$, $\alpha^j \in \mathbb{R}^{d_{\mathscr{L}}}$, $j=1,\ldots,r$, then $v_{2m}(x) = x^{\{m\}\prime} V x^{\{m\}}$ is an HPLF for (1)–(3).

Proof. If (6) admits solution, then $v_{2m}(x) = x^{\{m\}\prime}Vx^{\{m\}}$ is positive definite. By differentiating $v_{2m}(x)$ along the trajectories of the system and exploiting properties (I) and (II) above, one gets

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} v_{2m}(x)\right|_{\dot{x}^{\{m\}}=\bar{A}_{j,\{m\}}x^{\{m\}}} = x^{\{m\}\prime}(V\bar{A}_{j,\{m\}} + \bar{A}'_{j,\{m\}}V)x^{\{m\}} = x^{\{m\}\prime}(V\bar{A}_{j,\{m\}} + \bar{A}'_{j,\{m\}}V + L(\alpha^j))x^{\{m\}}, \tag{7}$$

which is negative definite by (6). Observe that the CSMR (4) has been used in (7). Now, notice that, due to property (I), matrices $\bar{A}_{j,\{m\}} = (A_{0,\{m\}} + \sum_{i=1}^{s} w_i^j A_{i,\{m\}})$ are the vertices of the set of extended matrices $A_{0,\{m\}} + \sum_{i=1}^{s} w_i(t) A_{i,\{m\}}$, $w(t) \in \mathscr{W}$. Convexity of this set of matrices implies that $v_{2m}(x)$ is a common HPLF for all matrices of the set. Hence, property (II) allows one to conclude that $v_{2m}(x)$ is also an HPLF for system (1). □

It is worth observing that (6) is an LMI feasibility problem (Boyd et al., 1994), which can be solved by efficient computational tools based on convex optimization (Nesterov & Nemirovsky, 1993; Gahinet, Nemirovski, Laub, & Chilali, 1995). Notice that the free variables in (6) are matrix $V = V' \in \mathbb{R}^{d\times d}$ and vectors $\alpha^j \in \mathbb{R}^{d_{\mathscr{L}}}$, $j=1,\ldots,r$, and therefore their number is equal to $d(d+1)/2 + rd_{\mathscr{L}} - 1$ (the term -1 is due to the fact that V can be arbitrarily scaled).

3.2. Necessary and sufficient condition

In some cases, depending on the dimension n of x and the degree $2m$ of $v_{2m}(x)$, the sufficient condition provided by Theorem 1 is also necessary for the existence of an HPLF for (1)–(3). The next result is central for establishing such a necessary and sufficient condition.

Lemma 1. *Consider the set*

$$\mathscr{E} = \{(n,2), n\in\mathbb{N}\} \cup \{(2,2m), m\in\mathbb{N}\} \cup \{(3,4)\}, \tag{8}$$

where $\mathbb{N}$ denotes the set of positive integers. Let $x\in\mathbb{R}^n$ and $g_{2m}(x)$ be a nonnegative homogeneous form of degree $2m$. If $(n,2m)\in\mathscr{E}$, then $g_{2m}(x)$ can be written as the sum of squares of homogeneous forms of degree m. Moreover, if $g_{2m}(x)$ is positive, then it admits a positive definite SMR, i.e. there exists a positive definite matrix $V\in\mathbb{R}^{d\times d}$ such that $g_{2m}(x) = x^{\{m\}\prime}Vx^{\{m\}}$.

Proof. See Appendix A. □

Theorem 2. *Let $(n,2m)$ belong to $\mathscr{E}$ in (8). Then, there exists an HPLF of degree $2m$ for system (1) if and only if the set of LMIs (6) admits a feasible solution.*

Proof. Obviously, it must be proven only that if $(n,2m)\in\mathscr{E}$ and there exists an HPLF $v_{2m}(x)$ for (1), then (6) admits a feasible solution. By assumption we have that $v_{2m}(x)$ is positive definite, and $-\dot{v}_{2m}(x)$ is positive definite for any uncertainty $w(t)\in\mathscr{W}$. Hence, due to Lemma 1, there exists $V>0$ such that $v_{2m}(x) = x^{\{m\}\prime}Vx^{\{m\}}$, and there exist $V_j > 0$, $j=1,\ldots,r$, such that the time derivative of $v_{2m}(x)$ evaluated along the trajectories of $\dot{x}^{\{m\}} = \bar{A}_{j,\{m\}}x^{\{m\}}$ satisfies $-\dot{v}_{2m}(x) = x^{\{m\}\prime}V_jx^{\{m\}}$. Completeness of the CSMR parameterization (4) implies that there exists α^j such that $V_j = -V\bar{A}_{j,\{m\}} - \bar{A}'_{j,\{m\}}V - L(\alpha^j)$. Therefore (6) admits a feasible solution. □

3.3. Relationships with previous work on HPLF

The use of HPLFs for robust analysis of system (1)–(3) has been addressed in Zelentsovsky (1994). In that paper, a result based on the S-procedure is exploited to provide a sufficient condition for the existence of an HPLF of degree $2m$. Although the condition is formulated as the minimization of a nondifferentiable convex function, it can be easily rewritten as an LMI feasibility problem of the form

$$V > 0,$$
$$-V\bar{A}_{j,\{m\}} - \bar{A}'_{j,\{m\}}V - \sum_{i=1}^{d_Z}\beta_{ij}F_i > 0, \quad j=1,\ldots,r, \tag{9}$$

Table 1
Values of $d_{\mathscr{L}}$ (left) and d_Z (right) for some n and m

$m \backslash n$	2		3		4		5	
2	1	1	6	3	20	6	50	10
3	3	2	27	7	126	16	420	30
4	6	3	75	12	465	31	1990	65
5	10	4	165	18	1310	52	7000	121

where $F_i \in \mathbb{R}^{d\times d}$, $i=1,\ldots,d_Z$ are suitable matrices and $d_Z = d-n$. The difference between (9) and (6) is that the number of free variables β_{ij} for each LMI in (9) is d_Z, which turns out to be much smaller than $d_{\mathscr{L}}$, the dimension of vector α^j in (6), as clarified by Table 1.

The larger number of free variables in the LMIs makes the sufficient condition provided by Theorem 1 much more powerful. Moreover, the CSMR of homogeneous forms allows one to formulate the necessary and sufficient condition of Theorem 2 for the cases in which $(n,2m)\in\mathscr{E}$. On the contrary, the condition provided in Zelentsovsky (1994) is necessary only for the case $n=m=2$, in which the two parameterizations of homogeneous forms coincide (see Table 1).

4. Computation of the ℓ_∞ 2m-HPLF stability margin

In the analysis of uncertain systems of type (1)–(2), a key problem is that of finding the largest value of the positive scalar γ for which there exists an HPLF of degree $2m$ for all $w(t)$ belonging to the scaled perturbation set $\gamma\mathscr{W}$. In order to simplify the presentation, we restrict our attention to the case in which the perturbation set is the ℓ_∞ box, i.e.

$$\mathscr{B}_\gamma = \{q\in\mathbb{R}^s\colon |q_i| \leqslant \gamma,\ i=1,\ldots,s\}. \tag{10}$$

Clearly, $\mathscr{B}_\gamma$ is equal to $\gamma\,\mathrm{co}\{u^1,\ldots,u^{2^s}\}$, where u^j, $j=1,\ldots,2^s$, are the vertices of the unit ℓ_∞ ball $\mathscr{B}_1$. Said another way, the aim is to compute

$$\gamma_{2m}^* = \sup\{\gamma\colon \exists v_{2m}(x) \text{ for (1)–(2)},\ w(t)\in\mathscr{B}_\gamma\}. \tag{11}$$

In the following, γ_{2m}^* will be referred to as the ℓ_∞ $2m$-HPLF stability margin for system (1)–(2).

We also consider the related problem of computing the following quantity:

$$\kappa_{2m}^* = \sup\{\kappa\colon \exists v_{2m}(x) \text{ for (1)–(2)},\ w(t)\in\bar{\mathscr{B}}_\kappa\}, \tag{12}$$

$$\bar{\mathscr{B}}_\kappa = \{q\in\mathbb{R}^s\colon 0\leqslant q_i\leqslant\kappa,\ i=1,\ldots,s\}. \tag{13}$$

Notice that κ_{2m}^* differs from γ_{2m}^* because the perturbation set is restricted to the positive orthant of the parameter space. For this reason, it will be referred to as the ℓ_∞ $2m$-HPLF positive stability margin. It is worth recalling that the above stability margins γ_{2m}^* and κ_{2m}^* are bounded from above by the well-known ℓ_∞ state space parametric stability margin.

Let us first focus the attention on problem (11)–(10). Define the matrices $\tilde{A}_j = A(u^j)-A_0$, where u^j, $j=1,\ldots,2^s$ are the vertices of the unit ℓ_∞ ball $\mathscr{B}_1$. Moreover, let $\tilde{A}_{j,\{m\}}$ denote the extended matrix of $\tilde{A}_j$. The following result shows that a lower bound $\hat{\gamma}_{2m}^*$ of the ℓ_∞ $2m$-HPLF stability margin γ_{2m}^* can be computed by solving a quasiconvex optimization problem.

Theorem 3. *Let $\hat{\gamma}_{2m}^*$ be defined as*

$$(\hat{\gamma}_{2m}^*)^{-1} = \inf_{\substack{z\in\mathbb{R};\, V=V'\in\mathbb{R}^{d\times d};\\ \alpha^j\in\mathbb{R}^{d_{\mathscr{L}}},\ j=0,\ldots,2^s}} z$$

$$\text{s.t.}\ \begin{cases} V>0,\ z>0\\ -VA_{0,\{m\}} - A'_{0,\{m\}}V - L(\alpha^0) > 0\\ z(-VA_{0,\{m\}} - A'_{0,\{m\}}V - L(\alpha^0))\\ \quad > V\tilde{A}_{j,\{m\}} + \tilde{A}'_{j,\{m\}}V + L(\alpha^j)\ \ j=1,\ldots,2^s. \end{cases} \tag{14}$$

Then, $\hat{\gamma}_{2m}^*\leqslant\gamma_{2m}^*$. Moreover, if $(n,2m)$ belongs to $\mathscr{E}$, then $\hat{\gamma}_{2m}^*=\gamma_{2m}^*$.

Proof. Let $v_{2m}(x) = x^{\{m\}\prime}Vx^{\{m\}}$. The time derivative of $v_{2m}(x)$ evaluated for $w(t)=z^{-1}u^j$, is given by

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}v_{2m}(x)\right|_{w(t)=z^{-1}u^j}\\ &= x^{\{m\}\prime}[V(A_{0,\{m\}} + z^{-1}\tilde{A}_{j,\{m\}})\\ &\quad + (A_{0,\{m\}} + z^{-1}\tilde{A}_{j,\{m\}})'V]x^{\{m\}}\\ &= z^{-1}x^{\{m\}\prime}[z(VA_{0,\{m\}} + A'_{0,\{m\}}V + L(\alpha^0))\\ &\quad + V\tilde{A}_{j,\{m\}} + \tilde{A}'_{j,\{m\}}V + L(\alpha^j)]x^{\{m\}}, \end{aligned}$$

where the CSMR has been exploited by adding the term $zL(\alpha^0)+L(\alpha^j)$ inside the square brackets. Hence, the constraint in (14) guarantees that $\dot{v}_{2m}(x)$ is negative definite. Since this holds for all $w(t)$ such that $\|w(t)\|_\infty\leqslant z^{-1}$, one has that $\hat{\gamma}_{2m}^*\leqslant\gamma_{2m}^*$. Let us assume now that $(n,2m)$ belongs to $\mathscr{E}$. From definition (11), for any z such that $z^{-1}<\gamma_{2m}^*$, there exists a positive definite $v_{2m}(x)$ such that $-\dot{v}_{2m}(x)$ is positive definite for $w(t)$: $\|w(t)\|_\infty\leqslant z^{-1}$. From Lemma 1, it follows that $-\dot{v}_{2m}(x)$ admits a positive definite SMR matrix for any $w(t)\in\mathscr{B}_{z^{-1}}$ and therefore the constraint in problem (14) admits a feasible solution for any $z^{-1}<\gamma_{2m}^*$. Hence $\hat{\gamma}_{2m}^*=\gamma_{2m}^*$. □

Problem (14) is a generalized eigenvalue problem (GEVP), which has been proven to be a quasiconvex optimization problem and can be tackled by efficient optimization tools (Boyd et al., 1994; Nesterov & Nemirovsky, 1993). Notice that the second LMI constraint in (14) guarantees that the matrix multiplying the generalized eigenvalue z is positive, as required by the standard form of the GEVP (see Boyd et al., 1994). Observe that the number of free variables in the GEVP (14) is equal to $d(d+1)/2+(2^s+1)d_{\mathscr{L}}$.

A result similar to Theorem 3 can be obtained for the ℓ_∞ $2m$-HPLF positive stability margin κ^*_{2m} in (12)–(13). Here, we report the result for the simpler case of a segment of matrices, because it has been widely addressed in the literature and it will be useful in the examples presented in the next section. The aim is to compute the largest value of κ such that there exists an HPLF of degree $2m$ for the matrices $A_0+w(t)A_1$, with $0 \leqslant w(t) \leqslant \kappa$.

Corollary 1. *Let $s=1$ in (12)–(13). Define $\hat{\kappa}^*_{2m}$ as*

$$(\hat{\kappa}^*_{2m})^{-1} = \min_{\substack{z\in\mathbb{R};V=V'\in\mathbb{R}^{d\times d};\\ \alpha^1,\alpha^2\in\mathbb{R}^{d_{\mathscr{L}}}}} z$$

$$\text{s.t.} \begin{cases} V>0, z>0 \\ -VA_{0,\{m\}}-A'_{0,\{m\}}V-L(\alpha^1)>0 \\ z(-VA_{0,\{m\}}-A'_{0,\{m\}}V-L(\alpha^1)) \\ \quad > VA_{1,\{m\}}+A'_{1,\{m\}}V+L(\alpha^2). \end{cases} \tag{15}$$

Then, $\hat{\kappa}^*_{2m} \leqslant \kappa^*_{2m}$. Moreover, if $(n,2m)$ belongs to $\mathscr{E}$, then $\hat{\kappa}^*_{2m}=\kappa^*_{2m}$.

5. Construction of the optimal performance HPLF

Another problem of interest in the robustness analysis of uncertain systems is that of determining the Lyapunov function that achieves the best transient performance (see, for example, Olas, 1994). For a given Lyapunov function $v(x)$, one can define the transient performance index of system (1)–(3) as

$$\lambda(v) = \sup_{x\in\mathbb{R}^n\setminus 0}\ \sup_{w(t)\in\mathscr{W}} \frac{\dot{v}(x)}{v(x)}. \tag{16}$$

From (16) one has $v(x(t)) \leqslant v(x(t_0))e^{\lambda(v)(t-t_0)}$, thus establishing the rate of decrease of v. Therefore, it is natural to select among all feasible Lyapunov functions, the one that minimizes $\lambda(v)$. For systems of the type (1)–(3), this problem has been addressed in Olas (1994) within the class of quadratic Lyapunov functions. Here, the aim is to select the optimal Lyapunov function among HPLFs of degree $2m$, i.e. to compute

$$\lambda^*_{2m} = \inf_{v_{2m}} \lambda(v_{2m}). \tag{17}$$

The following result shows that an upper bound of λ^*_{2m} can be obtained by solving a GEVP.

Theorem 4. *Let $\hat{\lambda}^*_{2m}$ be defined as*

$$\hat{\lambda}^*_{2m} = \inf_{\substack{z\in\mathbb{R};V=V'\in\mathbb{R}^{d\times d};\\ \alpha^j\in\mathbb{R}^{d_{\mathscr{L}}},\ j=1,\ldots,r}} z$$

$$\text{s.t.} \begin{cases} V>0, \\ zV > V\bar{A}_{j,\{m\}}+\bar{A}'_{j,\{m\}}V+L(\alpha^j) \quad j=1,\ldots,r. \end{cases} \tag{18}$$

Then, $\hat{\lambda}^*_{2m} \geqslant \lambda^*_{2m}$. Moreover, if $(n,2m)$ belongs to $\mathscr{E}$, then $\hat{\lambda}^*_{2m}=\lambda^*_{2m}$.

Proof. Let us assume that there exists $z\in\mathbb{R}$ and $V=V'\in\mathbb{R}^{d\times d}$ satisfying the constraint in (18), for some vectors $\alpha^j\in\mathbb{R}^{d_{\mathscr{L}}}$. Then, by setting $v_{2m}(x)=x^{\{m\}\prime}Vx^{\{m\}}$ and differentiating along the trajectories of $\dot{x}^{\{m\}}=\bar{A}_{j,\{m\}}x^{\{m\}}$, one has

$$\left.\frac{\dot{v}_{2m}(x)}{v_{2m}(x)}\right|_{\dot{x}^{\{m\}}=\bar{A}_{j,\{m\}}x^{\{m\}}} = \frac{x^{\{m\}\prime}(V\bar{A}_{j,\{m\}}+\bar{A}'_{j,\{m\}}V)x^{\{m\}}}{x^{\{m\}\prime}Vx^{\{m\}}} = \frac{x^{\{m\}\prime}(V\bar{A}_{j,\{m\}}+\bar{A}'_{j,\{m\}}V+L(\alpha^j))x^{\{m\}}}{x^{\{m\}\prime}Vx^{\{m\}}} \leqslant z$$

for all $j=1,\ldots,r$. Therefore, by exploiting convexity of $\mathscr{W}$, one can conclude that $\dot{v}_{2m}/v_{2m}<z$ for all $w(t)\in\mathscr{W}$ and hence $\lambda^*_{2m}\leqslant z$. Minimizing with respect to z, one obtains the upper bound $\hat{\lambda}^*_{2m}$. Now, let $(n,2m)\in\mathscr{E}$. Definition (17) says that for any $z>\lambda^*_{2m}$ there exists a positive definite $v_{2m}(x)$ such that $\dot{v}_{2m}(x)/v_{2m}(x)<z$, for all x and for all $w(t)\in\mathscr{W}$. Then, Lemma 1 implies that $zv_{2m}(x)-\dot{v}_{2m}(x)$ admits a positive definite SMR matrix for any $w(t)\in\mathscr{W}$, and therefore the constraint in problem (18) admits a feasible solution for any $z>\lambda^*_{2m}$. □

6. Examples

Example 1. This example is taken from Zelentsovsky (1994). Let us consider the matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

and assume we want to compute the ℓ_∞ $2m$-HPLF positive stability margin κ^*_{2m}, defined in (12)–(13). Since in this example $n=2$, the lower bound provided by the GEVP (15) is tight, i.e. $\hat{\kappa}^*_{2m}=\kappa^*_{2m}$, as assured by Corollary 1.

In Zelentsovsky (1994), it has been observed that a quadratic Lyapunov function exists only for $\kappa<3.82$, and that this bound can be improved to $\kappa<5.73$ by solving problem (9) with $m=2$. Table 2 shows the values of κ^*_{2m} obtained from (15), for some different m.

The same example has been considered also in Blanchini and Miani (1996) and Xie et al. (1997). In Blanchini and Miani (1996), a polyhedral Lyapunov function has been constructed, guaranteeing asymptotic stability for $\kappa=6$. In Xie et al. (1997), a piecewise quadratic Lyapunov function achieving stability for $\kappa=6.2$, has been obtained via a sequence of LMI optimizations and a grid search over two free

Table 2
Values of κ^*_{2m} for different m in Example 1

m	2	3	4	5	6	7	8
κ^*_{2m}	5.73	6.21	6.39	6.64	6.65	6.78	6.79

parameters. As it can be seen from Table 2, these bounds are improved by all the HPLFs of degree greater than 2. Note that the HPLF is obtained by solving one GEVP, with $\frac{1}{2}(3m^2+m+2)$ free variables.

Example 2. This example shows that the gap in the Lyapunov stability margin between our technique and the one presented in Zelentsovsky (1994) can be very large. Let us consider the matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -2 & 0 & -1 \\ 1 & -10 & 3 \\ 3 & -4 & 2 \end{pmatrix}$$

and assume we want to solve the same problem as in Example 1. Quadratic stability is guaranteed only for $\kappa \leqslant 1.9042$. Let us compute an HPLF of degree 4. Being $n=3$ and $m=2$ it follows that $d=6$ and $d_{\mathscr{L}}=6$. Then, vector $x^{\{2\}}$ and matrix $L(\alpha)$ are, respectively, given by $x^{\{2\}} = (x_1^2\ x_1x_2\ x_1x_3\ x_2^2\ x_2x_3\ x_3^2)$ and

$$L(\alpha) = \begin{pmatrix} 0 & 0 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 0 & 2\alpha_1 & \alpha_2 & 0 & -\alpha_4 & -\alpha_5 \\ 0 & \alpha_2 & 2\alpha_3 & \alpha_4 & \alpha_5 & 0 \\ -\alpha_1 & 0 & \alpha_4 & 0 & 0 & -\alpha_6 \\ -\alpha_2 & -\alpha_4 & \alpha_5 & 0 & 2\alpha_6 & 0 \\ -\alpha_3 & -\alpha_5 & 0 & -\alpha_6 & 0 & 0 \end{pmatrix}.$$

The GEVP (15) returns the lower bound $\hat{\kappa}_4^* = 75.1071$. Moreover, since $(n, 2m) = (3, 4) \in \mathscr{E}$, we have from Corollary 1 that $\hat{\kappa}_4^* = \kappa_4^*$.

Using the approach proposed in Zelentsovsky (1994), one finds that the maximum κ for which robust stability is guaranteed is equal to 17.8347. The remarkable difference with respect to our approach is due to the fact that the parameterization of homogeneous forms adopted in Zelentsovsky (1994) is not complete, as discussed in Section 3.3. Specifically, it is easy to see that in Zelentsovsky (1994) only the parameters $\alpha_1, \alpha_3, \alpha_6$ in the matrix $L(\alpha)$ are considered (see also Table 1).

Example 3. Let us consider the differential equation

$$\ddot{\xi}(t) + \dot{\xi}(t) + k(t)\xi(t) = 0$$

and assume that we want to compute the maximum κ such that the solution remains bounded, for all $0 \leqslant k(t) \leqslant \kappa$.

Table 3
Values of κ^*_{2m} for different m in Example 3

m	1	2	3	4	5	6
κ^*_{2m}	1.00	1.50	1.99	2.29	2.40	2.50
m	7	8	9	10	11	12
κ^*_{2m}	2.61	2.66	2.69	2.74	2.77	2.79

By adopting a strategy similar to that proposed in Brockett (1970), one can determine numerically the maximum value of κ. In particular, one has that such value must satisfy the equation

$$\sqrt{k} \exp\left\{-\frac{1}{\sqrt{4\kappa-1}}[\pi - \arctan(\sqrt{4\kappa-1})]\right\} - 1 = 0$$

and is equal to $\kappa^* \approx 3.0448$. Clearly, the problem can be easily tackled in the framework of HPLF by solving (12)–(13) with $s=1$ and

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The values of κ^*_{2m} obtained from (15) are reported in Table 3.

Notice that the values exhibit a growth towards the theoretical upper bound κ^*. On the other hand, the result in Blanchini and Miani (1999) guarantees that for any $\kappa < \kappa^*$ there exists an HPLF of suitable degree which is also a sum of squares; hence, it is expected that κ^*_{2m} converges to κ^* as m approaches infinity.

Another interesting byproduct of the treatment in Brockett (1970) is that one can calculate the worst-case sequence $k(t)$ (which consists of suitable switchings between $k(t)=0$ and $k(t)=\kappa^*$) and the corresponding trajectory of the system, that represents the limit curve towards which the level surfaces of the Lyapunov functions are expected to tend. This is confirmed by Fig. 1a, where the level curves of the obtained HPLFs for different values of m are depicted. Fig. 1b clearly shows that the HPLF corresponding to $m=12$ is very close to the limit trajectory predicted by the theory.

Example 4. This example has been considered by several authors (Radziszewski, 1977; Olas, 1994; Blanchini, 1995). Let us consider system (1)–(2) with $s=1$ and matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The maximum γ for which there exists a Lyapunov function for system (1)–(2), with $w(t) \in \mathscr{B}_\gamma$, has been calculated for different classes of Lyapunov functions. Notice that the ℓ_∞ state space stability margin is given by $\rho=1$, because A_0+A_1 is not asymptotically stable, and hence $\gamma^*_{2m} \leqslant 1$, for all m. In Radziszewski (1977), it has been proven that quadratic

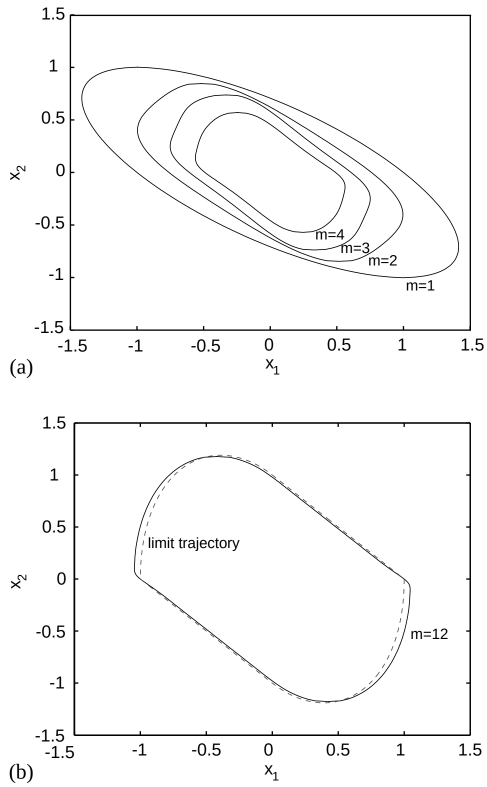

Fig. 1. Example 3: (a) Level curves of $v_{2m}(x)$ for different m; (b) level curve of $v_{24}(x)$ and limit trajectory (dashed).

Lyapunov functions can achieve $\gamma_2^* = \sqrt{3}/2$. In Blanchini (1995), a polyhedral Lyapunov function has been provided, which guarantees stability for $|w(t)| \leqslant \gamma = 0.98$. The level curves of this Lyapunov function have 30 vertices.

By applying Theorem 3 for different values of m, one can construct HPLFs of increasing degree, trying to improve the value of the $2m$-HPLF stability margin. Observe that also in this example the lower bound provided by (14) always coincides with γ_{2m}^* because $n = 2$. For $m = 2$, one obtains $\gamma_4^* = 0.9771$, but for $m \geqslant 3$ the result returned by the GEVP is $\hat{\gamma}_{2m}^* = 1$, which means that it is possible to construct an HPLF of degree 6 (or more) for $|w(t)| \leqslant \gamma$ and γ arbitrarily close to 1.

Example 5. This example concerns the computation of an optimal performance HPLF, in the sense explained in Section 5. Let us consider the helicopter model originally proposed in Narendra and Tripathi (1973), and the robust controller designed in Chen and Chen (1991). The resulting closed-loop uncertain system has four state variables and three uncertain parameters q_i satisfying $|q_1(t)| \leqslant 0.2192$, $|q_2(t)| \leqslant 1.2031$, $|q_3(t)| \leqslant 2.0673$. The system can be easily written in form (1)–(3) with $r = 8$.

In Olas (1994), the problem of computing the quadratic Lyapunov function achieving the best transient performance, defined as in (16), has been addressed. The value obtained was $\lambda = -0.3839$. By solving the GEVP problem (18) with $m = 2$, one obtains an HPLF of degree four achieving $\hat{\lambda}_4^* = -0.8889$. Therefore, once again it can be seen that the HPLFs outperform the classic approaches based on quadratic Lyapunov functions. Moreover, this example shows that the proposed LMI-based procedures are able to handle also quite complex systems (in this case, a fourth-order uncertain system with eight vertices).

7. Conclusions

The construction of HPLFs for linear systems with time-varying structured uncertainties has been addressed via LMI optimization techniques. With respect to previous work on this class of Lyapunov functions, better results have been obtained by exploiting a complete parameterization of homogeneous forms of given degree. Moreover, this allows one to formulate necessary conditions for the existence of an HPLF in some cases (for example, for all second-order systems). Comparisons with other classes of Lyapunov functions are also very promising.

Further investigations on the potentialities of HPLFs are currently under development. The use of this class of Lyapunov functions for systems with time-invariant structured uncertainties, and the possibility of employing parameter-dependent HPLFs to improve performances in robustness analysis will be the subject of future research. Another interesting topic concerns the role of HPLFs in robust control design procedures.

Appendix A. Proof of Lemma 1

The first part is a well-known property of homogeneous forms (Hardy, Littlewood, & Pólya, 1988). In order to prove the second part, let us define the normalized minimum of $g_{2m}(x)$ as

$$\varepsilon_g = \min_{x \in \mathbb{R}^n} \quad g_{2m}(x)$$
$$\text{s.t.} \quad \|x\| = 1.$$

Obviously, $\varepsilon_g > 0$ since $g_{2m}(x)$ is positive definite. Let us introduce the homogeneous form $h_{2m}(x) = g_{2m}(x) - \varepsilon_g \|x\|^{2m}$. It turns out that $h_{2m}(x)$ is a nonnegative homogeneous form. Indeed, its normalized minimum is equal to $\varepsilon_h = \varepsilon_g - \varepsilon_g = 0$. Hence, the first part of this lemma guarantees that $h_{2m}(x)$ can be expressed as sum of squares of polynomials if $(n, 2m) \in \mathscr{E}$, i.e.

$$h_{2m}(x) = \sum_i f_{m,i}^2(x) = \sum_i (t_i' x^{\{m\}})^2$$
$$= x^{\{m\}\prime} \left(\sum_i t_i' t_i \right) x^{\{m\}} = x^{\{m\}\prime} H x^{\{m\}},$$

where $f_{m,i}$ are homogeneous forms of degree m, $t_i \in \mathbb{R}^d$ and $H = \sum_i t_i' t_i \geqslant 0$. Therefore, there exists a positive semidefinite SMR matrix for $h_{2m}(x)$. Let us define the following SMR matrix G of $g_{2m}(x)$: $G = H + \varepsilon_g N$ where N is the diagonal SMR matrix of $\|x\|^{2m}$ (such N exists since the homogeneous form $\|x\|^{2m}$ contains only monomials with even powers of the variables x_i). Then, let us observe that $N \geqslant I_d$ since the coefficients of the monomials in $\|x\|^{2m}$ are greater or equal to 1. Therefore, $G \geqslant H + \varepsilon_g I_d > 0$, that is G is positive definite.

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