Constructive Provability Logic [1]

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Abstract

We present two variants of an intuitionistic sequent calculus that validates the Löb rule and permits logical reflection over provability. We explore properties of both variants of this logic, which we call *constructive provability logic* due to its close ties to Gödel-Löb provability logic.

Keywords: modal logic, provability logic, sequent calculus, intuitionistic logic

1 Introduction

In this paper, we describe and investigate *constructive provability logic*, an intuitionistic logic that admits reasoning about provability and non-provability. Constructive provability logic is closely tied to the provability logic **GL** [7]. As a structural proof theory for modal logic, constructive provability logic is also connected to Pfenning and Davies' judgmental reconstruction of **S4** [3] and to Simpson's intuitionistic Kripke semantics (**IK**) [6]; this previous work did not, however, consider provability logic.

Constructive provability logic also draws from work on logics with *definitional reflection* [4]. We will use definitional reflection over both the accessibility relation that underlies the modal logic and – most critically and unusually – definitional reflection *over the definition of the logic itself*.

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To lay the foundation for constructive provability logic, we present in Section 1.1 a simplified use of definitional reflection in a modal logic. In Section 1.2 we discuss reflection over logical provability, which gives us constructive provability logic. In Section 2 we present two variants of constructive provability logic, and in Section 3 we consider the relationship between this logic and classical Hilbert-style presentations of provability logic. For brevity, we limit our discussion to the sequent calculus; a separate technical report discusses the judgmental principles of constructive provability logic and connects the natural deduction and sequent calculus presentations [5].

1.1 Reflection over an accessibility relation

Kripke semantics for modal logic are characterized by *worlds* and an *accessibility relation* that describes the relationship between worlds. We will use as a running example an accessibility relation with three worlds, α, β, and γ, such that $\alpha \prec \beta$ (we say "β is accessible from α"), $\alpha \prec \gamma$, and $\beta \prec \gamma$.

We can sketch a proof theory for a modal logic **DML** (for "**D**efinitional reflection in **M**odal **L**ogic") parametrized over an arbitrary accessibility relation; the three-place accessibility relation above is one possible example. The hypothetical judgment for this logic takes the form $A_1[w_1], \ldots, A_n[w_n] \Rightarrow C[w]$, where C and the A_i are propositions and w and the w_i are worlds. Following the judgmental methodology [3], we state the logic's defining principles:

Defining principles of DML

- *Weakening principle*: If $\Gamma \subseteq \Gamma'$ and $\Gamma \Rightarrow A[w]$, then $\Gamma' \Rightarrow A[w]$.
- *Identity principle*: If $A[w] \in \Gamma$, then $\Gamma \Rightarrow A[w]$.
- *Cut principle*: If $\Gamma \Rightarrow A[w]$ and $\Gamma, A[w] \Rightarrow C[w']$, then $\Gamma \Rightarrow C[w']$.

These defining principles mirror the usual defining principles for hypothetical reasoning about truth in intuitionistic logics; the weakening principle encompasses the "structural rules" of contraction, exchange, and weakening.

In **DML**, as in Simpson's **IK**, the accessibility relation is critical to the definition of the modal operators. Consider the definition of modal possibility, $\Diamond A$. The Kripke interpretation of modal possibility is that $\Diamond A$ is true at world w if there exists some accessible world w' where A is true. The right sequent rule for modal possibility directly reflects this interpretation:

$$\frac{w \prec w' \quad \Gamma \Rightarrow A[w']}{\Gamma \Rightarrow \Diamond A[w]} \Diamond R$$

The left sequent rule for modal possibility is where the use of definitional reflection becomes important. If we have as an assumption that $\Diamond A$ is true at world w, we can use case analysis over the pre-defined accessibility relation to "look up" all the worlds w' such that $w \prec w'$ holds; for each such w', we must prove the ultimate conclusion using the additional assumption $A[w']$. This is expressed by the following inference rule:

$$\frac{\Diamond A[w] \in \Gamma \quad \forall w'.\, w \prec w' \longrightarrow \Gamma, A[w'] \Rightarrow C[w'']}{\Gamma \Rightarrow C[w'']}\ \Diamond L$$

In our aforementioned example, there are two worlds w' such that $\alpha \prec w'$ holds. Therefore, if we use a hypothesis of the form $\Diamond A[\alpha]$, we must consider the case where A holds at world β and the case where A holds at world γ. Similarly, because there are zero worlds w' such that $\gamma \prec w'$ holds, a hypothesis of the form $\Diamond A[\gamma]$ is immediately contradictory and can be used to prove anything at all. These two derivable special cases of the $\Diamond L$ rule can be written as follows:

$$\frac{\Diamond A[\alpha] \in \Gamma \quad \Gamma, A[\beta] \Rightarrow C[w''] \quad \Gamma, A[\gamma] \Rightarrow C[w'']}{\Gamma \Rightarrow C[w'']}\ \Diamond L_\alpha \quad \frac{\Diamond A[\gamma] \in \Gamma}{\Gamma \Rightarrow C[w'']}\ \Diamond L_\gamma$$

It is possible, at least in this simple case, to see $\Diamond L$ as merely a "rule schema" that, once given an accessibility relation, stamps out an appropriate number of rules. However, as suggested by Zeilberger [8], it is more auspicious to take this "higher-order formulation" of definitional reflection at face value: the second premise of the $\Diamond L$ rule is actually a mapping – a function – from facts about the accessibility relation to derivations.

1.2 Reflection over provability

The system **DML** was just a warm-up; we will now introduce constructive provability logic by additionally using definitional reflection over provability. In **DML**, a hypothetical assumption $\Diamond A[w] \in \Gamma$ allows us to assume (by the addition of a new hypothetical assumption) that A is true at one of the worlds w' accessible from w; if there is no such world w', the assumption is contradictory. In **CPL** (for "**C**onstructive **P**rovability **L**ogic"), we want a hypothetical assumption $\Diamond A[w] \in \Gamma$ to represent an assumption that A is *provable given the current set of hypotheses* at one of the worlds w' accessible from w. If A is not currently provable at some world w' accessible from w, the assumption is contradictory.

As a specific example, if Q is an arbitrary atomic proposition, $\bot$ is the proposition representing logical falsehood, and we use the accessibility relation from the previous section, then in **CPL** we can prove $\Diamond Q[\alpha] \Rightarrow \bot[\alpha]$. There is no **CPL** proof of $\Diamond Q[\alpha] \Rightarrow Q[\beta]$ and no proof of $\Diamond Q[\alpha] \Rightarrow Q[\delta]$, therefore, an assumption $\Diamond Q[\alpha]$ asserting that Q is currently provable at one of the worlds w' accessible from α is contradictory in **CPL**. The same sequent $\Diamond Q[\alpha] \Rightarrow \bot[\alpha]$ would *not* have been provable in **DML**. In order to use the hypothesis $\Diamond Q[\alpha]$ in **DML**, we would have to prove both $\Diamond Q[\alpha], Q[\beta] \Rightarrow \bot[\alpha]$ and $\Diamond Q[\alpha], Q[\delta] \Rightarrow \bot[\alpha]$, and neither of these are provable.

Reflection over provability must be done with care. It would be logically inconsistent to modify our previous left rule for modal possibility by turning the hypothesis $A[w']$ into a higher-order assumption $\Gamma \Rightarrow A[w']$ like this:

$$\frac{\Diamond A[w] \in \Gamma \quad \forall w'.\, w \prec w' \longrightarrow \Gamma \Rightarrow A[w'] \longrightarrow \Gamma \Rightarrow C[w'']}{\Gamma \Rightarrow C[w'']}\ \Diamond L_{bad}$$

This definition is inconsistent because the hypothetical judgment $\Gamma \Rightarrow A[w']$ occurs "negatively" – to the left of an arrow – in a rule that is ostensibly defining the hypothetical judgment. The first step in our solution is to modify the accessibility relation so that it is *converse well-founded* (no cycles or infinite ascending chains). This well-founded accessibility relation will enable us to define the hypothetical judgment one world at a time: when $w \prec w'$, then we will define $\Gamma \Rightarrow A[w']$ before $\Gamma \Rightarrow A[w]$ in the same way we defined the accessibility relation $w \prec w'$ before $\Gamma \Rightarrow A[w]$ in **DML**.

If we are trying to define provability one world at a time, the problem with $\Diamond L_{bad}$ is the relationship (or lack thereof) between $\Gamma \Rightarrow A[w']$, which we are reflecting over, and $\Gamma \Rightarrow C[w'']$, which we are defining. Because $w \prec w'$, the simplest solution is to force w to be equal to w''; this results in the following "tethered" (in the sense that the hypothesis $\Diamond A[w]$ is tethered to the conclusion $C[w]$) left rule for modal possibility:

$$\frac{\Diamond A[w] \in \Gamma \quad \forall w'.\, w \prec w' \longrightarrow \Gamma \Rightarrow A[w'] \longrightarrow \Gamma \Rightarrow C[w]}{\Gamma \Rightarrow C[w]}\ \Diamond L$$

In our previous technical report [5], we only considered this tethered version of constructed provability logic; we call this system **CPL**.

The tethered proof theory of **CPL** can be viewed as unnecessarily restrictive. To fix the inconsistent left rule $\Diamond L_{bad}$, all that is really necessary according to the discussion above is for provability at w' to be defined before provability at w''. We can therefore "de-tether" the logic somewhat by merely requiring that the world w be "ahead" of the world w'' in the reflexive, transitive closure of the accessibility relation (written as $w'' \prec^* w$). This is sufficient to ensure that w' will be "ahead" of w'' in the irreflexive, transitive closure of the accessibility relation (written as $w'' \prec^+ w'$), ensuring provability at w' can be defined before provability at w''. The de-tethered left rule for modal possibility in constructive provability logic looks like this:

$$\frac{w'' \prec^* w \quad \Diamond A[w] \in \Gamma \quad \forall w'.\, w \prec w' \longrightarrow \Gamma \Mapsto A[w'] \longrightarrow \Gamma \Mapsto C[w'']}{\Gamma \Mapsto C[w'']}\ \Diamond L$$

We call the de-tethered variant of constructive provability logic **CPL***. To distinguish the two similar logics, in the subsequent discussion we will write the hypothetical judgment for **CPL** as $\Gamma \Rightarrow A[w]$ and write the hypothetical judgment for **CPL*** as $\Gamma \Mapsto A[w]$.

1.3 A note on formalization

Constructive provability logic and its metatheory has been formalized in the Agda proof assistant, an implementation of the constructive type theory of Martin Löf [1]. This development is available from http://bitbucket.org/robsimmons/constructive-provability-logic.

With two exceptions, all of the results in this paper are fully verified by Agda. The most significant exception is that Agda cannot verify that rules such as $\Diamond L$ above avoid logical inconsistency. This is because the "positivity checker," which ensures that datatypes are not self-referential, does not understand the critical relationship between the logical rules and the converse well-founded accessibility relation. The result is that the positivity checker must be disabled when we encode the rules in Figures 1 and 2. This issue is discussed further in the technical report along with potential resolutions [5].

The second issue is that, due to the complexity of both the proof and the induction metric for de-tethered cut (Theorem 2.2), Agda runs out of memory and crashes when attempting to verify this proof. We hope that re-factoring or improvements to Agda will eventually make checking this proof possible.

2 Intuitionistic constructive provability logic

In this section, we will present the defining principles and sequent calculi for both **CPL** (Section 2.2) and **CPL*** (Section 2.3). Though the identity principle for both logics is the same as the identity principle for **DML**, the weakening and cut principles must be revised.

2.1 Revising the weakening principle

We can see that the weakening principle must be revised by returning to the example that we considered in Section 1.2: the sequent $\Diamond Q[\alpha] \Rightarrow \bot[\alpha]$ is provable in constructive provability logic. If the logic obeyed the "usual" weakening principle, we would expect $\Diamond Q[\alpha], Q[\beta] \Rightarrow \bot[\alpha]$ to also be provable. But this is not the case: $\Diamond Q[\alpha], Q[\beta] \Rightarrow Q[\beta]$ *is* provable (by *init*), so the assumption of $\Diamond Q[\alpha]$ is not contradictory in the "weakened" context.

This illustrates that constructive provability logic needs a different notion of "weakened context" to avoid this illegitimate form of weakening. The new weakening relation $\Gamma \subseteq_w \Gamma'$, parametrized by a world w, holds if:

(i) For all w' such that $w \prec^* w'$, $A[w'] \in \Gamma$ implies $A[w'] \in \Gamma'$, and

(ii) For all w' such that $w \prec^+ w'$, $A[w'] \in \Gamma'$ implies $A[w'] \in \Gamma$.

This parametrized notion of weakened contexts allow us to state a generalized weakening principle – if $\Gamma \subseteq_w \Gamma'$ and $\Gamma \Rightarrow A[w]$ then $\Gamma' \Rightarrow A[w]$ – that holds for both **CPL** and **CPL***.

$$\frac{}{\Gamma, Q[w] \Rightarrow Q[w]}\ \textit{init} \quad (Q \text{ is an atomic proposition}) \quad \frac{\bot[w] \in \Gamma}{\Gamma \Rightarrow C[w]}\ \bot L$$

$$\frac{\Gamma, A[w] \Rightarrow B[w]}{\Gamma \Rightarrow A \supset B[w]}\ {\supset}R \quad \frac{A \supset B[w] \in \Gamma \quad \Gamma \Rightarrow A[w] \quad \Gamma, B[w] \Rightarrow C[w]}{\Gamma \Rightarrow C[w]}\ {\supset}L$$

$$\frac{w \prec w' \quad \Gamma \Rightarrow A[w']}{\Gamma \Rightarrow \Diamond A[w]}\ \Diamond R \quad \frac{\forall w'.\, w \prec w' \longrightarrow \Gamma \Rightarrow A[w']}{\Gamma \Rightarrow \Box A[w]}\ \Box R$$

$$\frac{\Diamond A[w] \in \Gamma \quad \forall w'.\, w \prec w' \longrightarrow \Gamma \Rightarrow A[w'] \longrightarrow \Gamma \Rightarrow C[w]}{\Gamma \Rightarrow C[w]}\ \Diamond L$$

$$\frac{\Box A[w] \in \Gamma \quad (\forall w'.\, w \prec w' \longrightarrow \Gamma \Rightarrow A[w']) \longrightarrow \Gamma \Rightarrow C[w]}{\Gamma \Rightarrow C[w]}\ \Box L$$

Fig. 1. Sequent calculus for intuitionistic **CPL**

2.2 Tethered constructive provability logic (**CPL**)

The rules for the **CPL** sequent calculus are presented in Fig. 1. Implication, atomic propositions and falsehood are defined as per usual in sequent calculi. The rules for modal possibility have been discussed in Sections 1.1 and 1.2, leaving us to discuss modal necessity. Whereas modal possibility usually is thought of as having an "existential character" (there *exists* some accessible world where A is true), modal necessity has a "universal character" (at *every* accessible world, A is true). We conclude $\Box A$ at world w if we can show that for all worlds w' that are accessible from w, A is provable at w'; this is reflected in the $\Box R$ rule.

The universal character of modal necessity would suggest that we can *use* a hypothesis $\Box A[w]$ by exhibiting a world w' accessible from w and then assuming that A was provable there.

$$\frac{\Box A[w] \in \Gamma \qquad w \prec w' \qquad \Gamma \Rightarrow A[w'] \longrightarrow \Gamma \Rightarrow C[w]}{\Gamma \Rightarrow C[w]}\ \Box L'$$

Surprisingly, this rule is not logically complete in the presence of potentially infinite accessibility relations, so **CPL** uses a less intuitive third-order formulation of $\Box L$ shown in Fig. 1. The more intuitive rule is nevertheless derivable from the actual $\Box L$ rule, and this issue is discussed in more detail in the technical report [5].

Having presented our sequent calculus, we must now show that the system makes sense from a logical point of view. For sequent calculi, this means establishing the standard principles of cut, identity and weakening.

Theorem 2.1 (Metatheory of the CPL sequent calculus)

- *Weakening: If* $\Gamma \subseteq_w \Gamma'$ *and* $\Gamma \Rightarrow A[w]$*, then* $\Gamma' \Rightarrow A[w]$.
- *Identity: If* $A[w] \in \Gamma$*, then* $\Gamma \Rightarrow A[w]$.
- *Cut: If* $\Gamma \Rightarrow A[w]$ *and* $\Gamma, A[w] \Rightarrow C[w]$*, then* $\Gamma \Rightarrow C[w]$.

$$\frac{}{\Gamma, Q[w] \Rrightarrow Q[w]}\ init \quad (Q \text{ is an atomic proposition}) \qquad \frac{w' \prec^* w \quad \bot[w] \in \Gamma}{\Gamma \Rrightarrow C[w']}\ \bot L$$

$$\frac{\Gamma, A[w] \Rrightarrow B[w]}{\Gamma \Rrightarrow A \supset B[w]}\ {\supset}R \quad \frac{A \supset B[w] \in \Gamma \quad w' \prec^* w \quad \Gamma \Rrightarrow A[w] \quad \Gamma, B[w] \Rrightarrow C[w']}{\Gamma \Rrightarrow C[w']}\ {\supset}L$$

$$\frac{w \prec w' \quad \Gamma \Rrightarrow A[w']}{\Gamma \Rrightarrow \Diamond A[w]}\ \Diamond R \quad \frac{\forall w'.\, w \prec w' \longrightarrow \Gamma \Rrightarrow A[w']}{\Gamma \Rrightarrow \Box A[w]}\ \Box R$$

$$\frac{\Diamond A[w] \in \Gamma \quad w'' \prec^* w \quad \forall w'.\, w \prec w' \longrightarrow \Gamma \Rrightarrow A[w'] \longrightarrow \Gamma \Rrightarrow C[w'']}{\Gamma \Rrightarrow C[w'']}\ \Diamond L$$

$$\frac{\Box A[w] \in \Gamma \quad w'' \prec^* w \quad (\forall w'.\, w \prec w' \longrightarrow \Gamma \Rrightarrow A[w']) \longrightarrow \Gamma \Rrightarrow C[w'']}{\Gamma \Rrightarrow C[w'']}\ \Box L$$

Fig. 2. Sequent calculus for intuitionistic **CPL***

2.3 De-tethered constructive provability logic (**CPL**)

The sequent calculus rules for **CPL*** are presented in Fig. 2. The only difference from the rules of the previous section is that we no longer restrict the conclusion of left rules to be at the world w of the hypothesis, instead allowing it to be at a world w'', provided that $w'' \prec^* w$. Not unexpectedly, this formulation of the sequent calculus allows us to prove more statements in our logic (as seen in the next section).

Like in our treatment of the sequent calculus of the previous section, we established the standard meta-theoretic results of cut, identity and weakening. We found in the proof of these properties that an additional principle was needed, which we called *decut* due to its symmetry with the cut principle. Decut states that if we can prove that something is true, we are allowed to assume it. Note that this does not follow from the weakening principle: the weakening principle in constructive provability logic only allows us to add new assumptions at the current world, whereas decut allows us to add new assumptions at (transitively) accessible worlds.

Theorem 2.2 (Metatheory of the CPL* sequent calculus)

- *Weakening: If* $\Gamma \subseteq_w \Gamma'$ *and* $\Gamma \Rrightarrow A[w]$*, then* $\Gamma' \Rrightarrow A[w]$.
- *Identity: If* $A[w] \in \Gamma$ *then* $\Gamma \Rrightarrow A[w]$.
- *Cut: If* $w' \prec^* w$, $\Gamma \Rrightarrow A[w]$*, and* $\Gamma, A[w] \Rrightarrow C[w']$*, then* $\Gamma \Rrightarrow C[w']$.
- *Decut: If* $w' \prec^* w$, $\Gamma \Rrightarrow A[w]$*, and* $\Gamma \Rrightarrow C[w']$*, then* $\Gamma, A[w] \Rrightarrow C[w']$.

3 Axiomatic characterization

In this section, we discuss the axiomatic characterization of **CPL** and **CPL***. The theorems in this section suffice to show that Hilbert-style reasoning is *sound* with respect to the sequent calculus. For instance, we say $\neg\Diamond\bot$ [4] is

[4] $\neg A$ is the usual intuitionistic negation $A \supset \bot$

an axiom of **CPL*** because, for all accessibility relations W, for all worlds $w \in W$, and for all contexts Γ, we can prove $\Gamma \Mapsto \neg\Diamond\bot[w]$. Some axioms, like $\Box A \supset \Box\Box A$, only hold in general when the accessibility relation is transitive; these are indicated. For every claimed non-axiom in the following section we have given a counterexample. For instance, $Q[\alpha] \Rightarrow (\neg\Diamond Q \supset \Box\neg Q)[\alpha]$ is unprovable, so the classically true De Morgan axiom $\neg\Diamond A \supset \Box\neg A$ does not hold in **CPL**. Both proofs and counterexamples for **CPL** and **CPL*** can be found in `TetheredCPL/Axioms.agda` and in `DetetheredCPL/Axioms.agda` (respectively) in the accompanying code.

We do not have any results that establish the *completeness* of constructive provability logic with respect to a Hilbert-style presentation. A satisfactory notion of completeness seems difficult to even state; constructive provability logic lacks a notion of "validity" that is critical to the statement and proof of Hilbert completeness for Pfenning-Davies **S4** [2].

3.1 Intuitionistic propositional logic

All of the axioms of intuitionistic propositional logic are true in both variants of constructive provability logic.

Theorem 3.1 (Axioms of intuitionistic propositional logic) *The following are axioms of both variants of constructive provability logic:*

(I) $A \supset A$
(K) $A \supset B \supset A$
(S) $(A \supset B \supset C) \supset (A \supset B) \supset A \supset C$
$(\bot E)$ $\bot \supset A$

3.2 Intuitionistic modal logic

The *necessitation* rule, $\Vdash A$ implies $\Vdash \Box A$, is fundamental to modal logics:

Theorem 3.2 (Necessitation rule) *If A is an axiom of either variant of constructive provability logic, then $\Box A$ is as well. Specifically, if $\Gamma \Rightarrow A[w]$ for all w, then $\Gamma \Rightarrow \Box A[w']$ for all w'.*

The fact that the world w in the premise is different than the world w' in the conclusion is critical.

In addition, all classical modal logics are characterized by the axiom K, $\Box(A \supset B) \supset \Box A \supset \Box B$. It is less clear what other axioms characterize intuitionistic modal logic; some of the axioms of Simpson's **IK** hold in neither Pfenning-Davies **S4** nor in constructive provability logic.

Theorem 3.3 (Axioms of IK, Simpson's intuitionistic modal logic) *The following are axioms of both variants of constructive provability logic:*

$(K\Box)$	$\Box(A \supset B) \supset \Box A \supset \Box B$	
$(K\Diamond)$	$\Box(A \supset B) \supset \Diamond A \supset \Diamond B$	
$(\mathit{4}\Box)$	$\Box A \supset \Box\Box A$	*(if the accessibility relation is transitive)*

The following are axioms of **CPL***:

$(\Diamond\bot)$	$\neg\Diamond\bot$	
$(\mathit{4}\Diamond)$	$\Diamond\Diamond A \supset \Diamond A$	*(if the accessibility relation is transitive)*

Axiom $\Diamond\bot$ is not an axiom of **CPL**, *and axiom IK, $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$, is not an axiom of either variant.*

If the accessibility relation is transitive, **CPL*** admits the axioms of Pfenning-Davies **S4**, plus $(\Diamond\bot)$, which holds in **IK** but not in Pfenning-Davies **S4**. We have not been able establish the status of axiom $\mathit{4}\Diamond$ in **CPL**.

Simpson's thesis presents axioms characterizing other properties of accessibility relations besides transitivity, but all these properties (e.g. symmetry) are inconsistent with converse well-foundedness, so we ignore them here.

3.3 Provability logic

Exploring the connection between constructive provability logic and provability logic was one of the motivations of this work. The most common characterization of provability logic is the *GL* axiom. Since *GL* can be used to prove the $\mathit{4}\Box$ axiom [7], it is not surprising that this axiom requires a transitive accessibility relation.

Theorem 3.4 (Axiom ***GL*****)** *If the accessibility relation is transitive, then axiom GL, $\Box(\Box A \supset A) \supset \Box A$, is an axiom of both variants of constructive provability logic.*

The other standard characterization of provability logic is the Löb rule. The Löb rule is almost always presented together with axiom $\mathit{4}\Box$ ensuring transitivity of the accessibility relation, but it is interesting to observe that the Löb rule, unlike the *GL* axiom, holds even without a transitive accessibility relation.

Theorem 3.5 (Löb rule) *If $\Box A \supset A$ is an axiom of either variant of constructive provability logic, then so is A. Specifically, if $\Gamma \Rightarrow \Box A \supset A[w]$ for all w, then $\Gamma \Rightarrow A[w']$ for all w'.*

Both theorems in this section are proved by induction over the accessibility relation; these are the only results in Section 3 that are proved by induction over the accessibility relation.

3.4 De Morgan laws

The interaction between negation and the modal operators is frequently an interesting ground for exploration. In classical modal logic, $\Diamond A$ is just defined as $\neg\Box\neg A$, and so all four of the De Morgan laws – $(\Diamond\neg A \supset \neg\Box A)$, $(\Box\neg A \supset \neg\Diamond A)$, $(\neg\Diamond A \supset \Box\neg A)$, and $(\neg\Box A \supset \Diamond\neg A)$ – hold trivially. The first three hold in Simpson's **IK**, and none hold in Pfenning-Davies **S4**. In **CPL*** two of the four hold, and in **CPL** the same two hold only if we make certain assumptions about consistency at accessible worlds.

Theorem 3.6 (De Morgan laws)

- *In* **CPL***, $\Diamond\neg A \supset \neg\Box A$ *and* $\Box\neg A \supset \neg\Diamond A$ *are axioms.*
- *In* **CPL**, *neither* $\Diamond\neg A \supset \neg\Box A$ *nor* $\Box\neg A \supset \neg\Diamond A$ *are axioms.*
- *In* **CPL**, *both* $\Gamma \Rightarrow \Diamond\neg A \supset \neg\Box A[w]$ *and* $\Gamma \Rightarrow \Box\neg A \supset \neg\Diamond A[w]$ *are true if there is no* $w \prec w'$ *such that* $\Gamma \Rightarrow \bot[w']$.
- $\neg\Diamond A \supset \Box\neg A$ *is not an axiom of* **CPL** *or* **CPL***.
- $\neg\Box A \supset \Diamond\neg A$ *is not an axiom of* **CPL** *or* **CPL***.

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